Densities of label-d cells in Fillomino patterns: exact values and constructions for $d \leq 25$

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1 The problem (TAOCP 7.2.2.3, exercise 296)

A *Fillomino* labelling assigns a positive integer to every cell of the infinite square grid $\mathbb{Z}^2$ so that, for every cell, the maximal rook-connected (edge-adjacent) region of equally labelled cells containing it has size (number of cells) exactly equal to that label. Thus label-1 cells are isolated singletons, label-2 cells form dominoes, label-k cells form k-ominoes, and two *distinct* maximal regions with the same label are never edge-adjacent.

For a fixed integer $d \geq 1$ let

$$\delta_d \;=\; \sup_F \; \overline{\mathrm{dens}}\big(\{\text{cells with label } = d\}\big),$$

the supremum, over all plane Fillomino labellings F, of the upper density of the label-d cells. Exercise 296 tabulates, for each $d \leq 25$, a lower bound (an explicit construction) and an upper bound, and asks whether these are the true values or whether some can be increased.

The upper bound throughout is the *halo* bound $\delta_d \leq d/(d + h_d)$ of the exercise: edge-disjoint polyominoes have disjoint halos, so an $n \times n$ pattern with k d-ominoes satisfies $k(d+h_d) \leq n^2+O(n)$, where $h_d = \lceil \sqrt{8d-4}\,\rceil/2$ is the minimal amortized halo cost of a d-omino (a half-integer in general, not a count of cells). Every construction we exhibit is a periodic pattern on a skew lattice that we have verified *wrap-free*: lifting one fundamental domain to $\mathbb{Z}^2$, every monochromatic rook-component has size exactly equal to its label, with no wrap-around.

2 The bounds of exercise 296

Table 1 reproduces the lower and upper bounds tabulated in the exercise. The six values for which the two already coincide—so that δ_d is determined—are shown in **green**.

3 Our results

We determine δ_d for 13 further values of d, and improve the lower bound for four more. The contributions are of two kinds: *combinatorial upper bounds* for $d = 1, 2$ (Section 3.1), which meet the book's lower bounds; and *explicit constructions* that raise the lower bound to the halo bound—thereby determining δ_d—for $d \in \{4, 7, 9, 12, 14, 16, 17, 19, 22, 23, 24\}$, together with strict but non-closing improvements for $d \in \{3, 6, 10, 20\}$ (Section 3.2).

d	lower	upper	d	lower	upper	d	lower	upper
1	3/8	1/2	10	2/3	20/29	19	19/27	38/51
2	4/9	1/2	11	**11/16**	**11/16**	20	5/7	40/53
3	1/2	6/11	12	2/3	12/17	21	3/4	42/55
4	8/15	4/7	13	**13/18**	**13/18**	22	22/31	22/29
5	**5/8**	**5/8**	14	7/10	28/39	23	23/32	23/30
6	3/5	12/19	15	5/7	30/41	24	3/4	24/31
7	7/12	7/11	16	16/23	8/11	25	**25/32**	**25/32**
8	**2/3**	**2/3**	17	17/24	17/23			
9	9/14	2/3	18	**3/4**	**3/4**			

Table 1: Lower and upper bounds tabulated in exercise 296 (the upper bound being the halo bound $d/(d+h_d)$). Green: the six values ($d \in \{5, 8, 11, 13, 18, 25\}$) where the two coincide, so δ_d is determined.

3.1 Combinatorial upper bounds for $d = 1$ and $d = 2$

For these two values the book's upper bound is the slack 1/2; we sharpen it combinatorially to meet the construction.

$d = 1$**:** $\delta_1 \leq 3/8$**.**

Label-1 cells are isolated singletons, hence form an *independent set*; every non-1 cell lies in a component of size ≥ 2. Each label-1 cell sends all four of its edges to non-1 cells, so the number of 1/non-1 edges is $E = 4N_1$, where N_1 is the number of label-1 cells. Grouping these edges by the non-1 component K they touch, $E = \sum_K e_K$, where e_K counts edges from K to label-1 cells (whose endpoints on the 1-side form an independent set).

Lemma. $e_K \leq \frac{12}{5}s$, where $s = |K|$.
(a) $s \geq 5$. Every such edge is a boundary edge of K, so $e_K \leq \mathrm{perim}(K) = 4s - 2a$, where a is the number of internally adjacent cell pairs of K. As K is connected, $a \geq s - 1$, hence $\mathrm{perim}(K) \leq 2s + 2 \leq \frac{12}{5}s$ for $s \geq 5$. (Holes only decrease the perimeter.)
(b) $s \leq 4$. Here $2s+2 > \frac{12}{5}s$, so we use that the 1-neighbours form an independent set. Assign each outer neighbour of K a weight equal to the number of edges it sends to K (a concave-corner cell has weight 2, others weight 1); then e_K is the maximum weight of a set of pairwise non-adjacent outer neighbours. A finite check of the one domino, two trominoes and five tetrominoes gives $e_K \leq 4, 6, 9$ for $s = 2, 3, 4$, the worst case $s = 4$ being the T-tetromino. All three are $< \frac{12}{5}s = 4.8,\ 7.2,\ 9.6$. $\square$

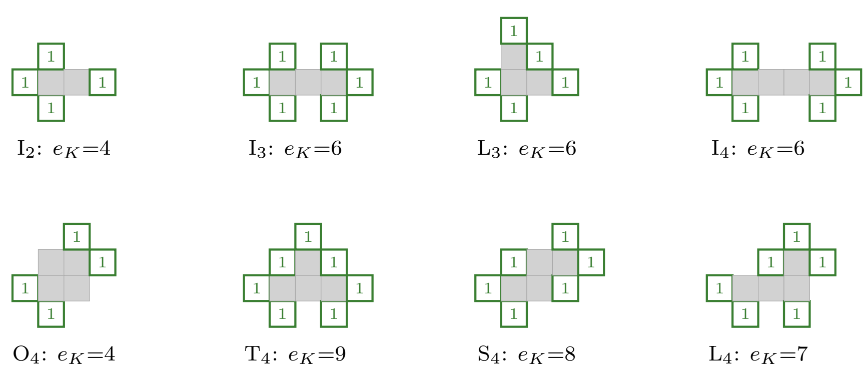


Figure 1: The eight free polyominoes with $|K| \le 4$ (grey $= K$; the green cells, each labelled 1, are a maximum-weight independent set of label-1 neighbours realizing e_K, a concave-corner neighbour counting twice). The worst ratio is the T-tetromino, $e_K = 9 < \frac{12}{5} \cdot 4$. The unique equality $e_K = \frac{12}{5}s$ occurs at $s = 5$ (the plus-pentomino, $e_K = 12$).

Summing over the non-1 components of an $n \times n$ torus (n^2 cells, no boundary, so $E = 4N_1$ exactly and every component is whole), $4N_1 = E = \sum_K e_K \le \frac{12}{5} \sum_K s = \frac{12}{5}(n^2 - N_1)$; hence $20N_1 \le 12(n^2 - N_1)$, i.e. $N_1/n^2 \le 3/8$. On an $n \times n$ window of the plane the same count changes by only $O(n)$ (edge cells, and components cut by the boundary), which vanishes after dividing by n^2; so $\delta_1 \le 3/8$. The book's construction (non-1 cells filled by label-5 pentominoes) attains 3/8, so $\boxed{\delta_1 = 3/8}$.

$d = 2$: $\delta_2 \le 4/9$.

Call a cell *black* if its label is 2, else *white*; let m be the number of dominoes and z the number of white cells.

Fact A. Black cells form an *induced matching* of dominoes: each label-2 region is a domino, and two dominoes are never edge-adjacent (they would merge into a size-4 region).

Fact B. No white component has size 2: a 2-cell white region could only be a label-2 domino (excluded, it is white) or two adjacent label-1 cells (which merge into a size-2 region), both impossible.

Marking. A horizontal domino has two white cells directly above it and two directly below (no adjacent domino), giving a *white–white edge* above and below; mark both. A vertical domino marks the white–white edges immediately to its left and right. Each domino contributes 2 marks, so the total number of marks is $2m$. A horizontal white–white edge can be marked only by the horizontal dominoes directly above/below it, and a vertical one only by the vertical dominoes to its left/right, so each white–white edge is marked at most twice. Writing a_i for the number of white–white edges marked i times,

$$2m = a_1 + 2a_2, \qquad e_{00} := \#\{\text{white–white edges}\} = a_0 + a_1 + a_2.$$

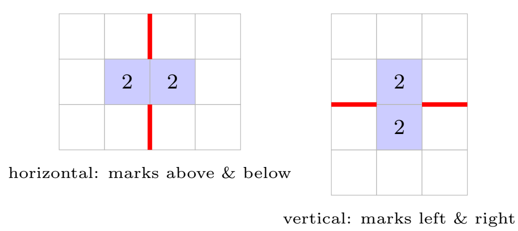
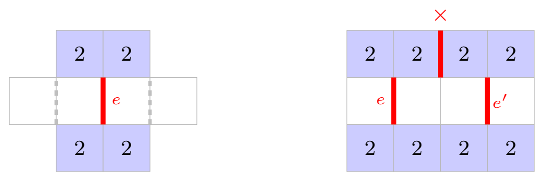


Figure 2: The marking. Each domino (blue) marks the white–white edge (red) on each of its two long sides; a white–white edge can receive at most two marks.

Lemma. $a_2 \le a_0$ (hence $e_{00} \ge a_1 + 2a_2 = 2m$).

Proof. By the marking rule the horizontal and vertical analyses are independent and symmetric; we treat horizontal edges row by row. Let e be a doubly-marked horizontal edge, with a horizontal domino directly above and directly below it. *(i)* Its left and right neighbour edges in the row, if white–white, are unmarked: a domino that would mark such a neighbour would have to occupy a cell already used by one of e's two dominoes. *(ii)* Two doubly-marked edges in a row cannot be exactly two apart: their two upper dominoes would then be edge-adjacent in the row above, contradicting Fact A. Hence doubly-marked edges in a row are pairwise at distance ≥ 3 and each is flanked by unmarked edges; mapping each to a private flanking unmarked edge is injective (the sole degenerate case—a length-2 white run whose single edge is doubly marked—is excluded by Fact B, as its two cells then form a size-2 white component). Thus $a_2^{\mathrm{hor}} \le a_0^{\mathrm{hor}}$, and likewise for columns, giving $a_2 \le a_0$. $\square$



(i) e (red) doubly marked; its neighbour edges (grey) can't be marked

(ii) distance 2: the two upper dominoes touch at $\times$ and would merge (Fact A)

Figure 3: The lemma. (i) A doubly-marked edge e (red) lies between two dominoes; its neighbouring white–white edges (grey) cannot be marked, since a marking domino would reuse a cell of e's dominoes. (ii) Two doubly-marked edges two apart would force their upper dominoes to be edge-adjacent ($\times$), merging into a size-4 region—excluded by Fact A.

Edge count. On an $n \times n$ torus, with $2m + z = n^2$ cells, the $2n^2$ edges split into black–black (m, one per domino), black–white ($6m$, six per domino) and white–white (e_{00}), so $2n^2 = 7m + e_{00} \ge 9m$. With $2m + z = n^2$ this gives $2z \ge 5m$, whence

$$\frac{2m}{2m+z} \le \frac{2m}{2m + \frac{5}{2}m} = \frac{4}{9}.$$

On an $n \times n$ window of the plane the same count incurs only $O(n)$ boundary corrections, and dividing by n^2 gives $\delta_2 \le 4/9$ as $n \to \infty$. The periodic pattern

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

(tiled by $3\mathbb{Z}^2$) attains $4/9$, so $\boxed{\delta_2 = 4/9}$.

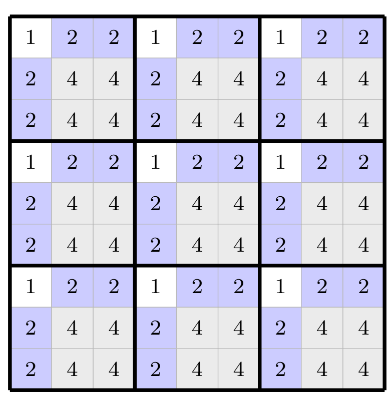


Figure 4: The lower-bound construction for $d = 2$: the 3×3 block above, tiled by $3\mathbb{Z}^2$ (thick lines mark the block boundaries). Label-2 cells (blue) form dominoes, label-4 cells form 2×2 squares, label-1 cells are singletons; 4 of every 9 cells carry label 2.

3.2 Lower-bound constructions

In each figure the label-d cells are shaded blue and the thick lines outline one repeating unit. For $d \in \{4, 7, 9, 12, 14, 16, 17, 19, 22, 23, 24\}$ the density equals the halo bound, so δ_d is determined; for $d \in \{3, 6, 10, 20\}$ it improves the book's lower bound.

$d = 4$**:** $\delta_4 = 4/7$ **(halo).**

4	1	4	4	2	2	4	4	1	4	4
4	4	2	2	4	4	1	4	4	2	2
2	2	4	4	1	4	4	2	2	4	4
4	4	1	4	4	2	2	4	4	1	4
1	4	4	2	2	4	4	1	4	4	2
4	2	2	4	4	1	4	4	2	2	4
2	4	4	1	4	4	2	2	4	4	1
4	1	4	4	2	2	4	4	1	4	4
4	4	2	2	4	4	1	4	4	2	2
2	2	4	4	1	4	4	2	2	4	4
4	4	1	4	4	2	2	4	4	1	4

$d = 7$**:** $\delta_7 = 7/11$ **(halo).**

7	7	1	7	7	7	7	2	2	7	1	7	7
2	2	7	1	7	7	1	7	7	7	7	2	2
7	7	7	7	2	2	7	1	7	7	1	7	7
1	7	7	1	7	7	7	7	2	2	7	1	7
7	2	2	7	1	7	7	1	7	7	7	7	2
1	7	7	7	7	2	2	7	1	7	7	1	7
7	1	7	7	1	7	7	7	7	2	2	7	1
7	7	2	2	7	1	7	7	1	7	7	7	7
7	1	7	7	7	7	2	2	7	1	7	7	1
2	7	1	7	7	1	7	7	7	7	2	2	7
7	7	7	2	2	7	1	7	7	1	7	7	7
7	7	1	7	7	7	7	2	2	7	1	7	7
2	2	7	1	7	7	1	7	7	7	7	2	2

$d = 9$: $\delta_9 = 2/3$ **(halo).**

2	2	9	1	9	1	9	9	9	9	2	9	9	9	1	9	9	9	2
9	9	2	9	9	9	1	9	9	9	2	9	9	9	9	1	9	1	9
9	9	2	9	9	9	9	1	9	1	9	2	2	9	1	9	1	9	9
9	1	9	2	2	9	1	9	1	9	9	9	9	2	9	9	9	1	9
1	9	9	9	9	2	9	9	9	1	9	9	9	2	9	9	9	9	1
9	1	9	9	9	2	9	9	9	9	1	9	1	9	2	2	9	1	9
9	9	1	9	1	9	2	2	9	1	9	1	9	9	9	9	2	9	9
9	1	9	1	9	9	9	9	2	9	9	9	1	9	9	9	2	9	9
2	9	9	9	1	9	9	9	2	9	9	9	9	1	9	1	9	2	2
2	9	9	9	9	1	9	1	9	2	2	9	1	9	1	9	9	9	9
9	2	2	9	1	9	1	9	9	9	9	2	9	9	9	1	9	9	9
9	9	9	2	9	9	9	1	9	9	9	2	9	9	9	9	1	9	1
9	9	9	2	9	9	9	9	1	9	1	9	2	2	9	1	9	1	9
1	9	1	9	2	2	9	1	9	1	9	9	9	9	2	9	9	9	1
9	1	9	9	9	9	2	9	9	9	1	9	9	9	2	9	9	9	9
9	9	1	9	9	9	2	9	9	9	9	1	9	1	9	2	2	9	1
9	9	9	1	9	1	9	2	2	9	1	9	1	9	9	9	9	2	9
2	9	1	9	1	9	9	9	9	2	9	9	9	1	9	9	9	2	9
9	2	9	9	9	1	9	9	9	2	9	9	9	9	1	9	1	9	2

$d = 12$: $\delta_{12} = 12/17$ **(halo).**

12	1	12	1	12	12	12	2	12	12	12	12	2	12	12	12
12	12	2	12	12	12	12	2	12	12	12	1	12	1	12	1
12	12	2	12	12	12	1	12	1	12	1	12	12	12	2	12
12	1	12	1	12	1	12	12	12	2	12	12	12	12	2	12
1	12	12	12	2	12	12	12	12	2	12	12	12	1	12	1
12	12	12	12	2	12	12	12	1	12	1	12	1	12	12	12
12	12	12	1	12	1	12	1	12	12	12	2	12	12	12	12
1	12	1	12	12	12	2	12	12	12	12	2	12	12	12	1
12	2	12	12	12	12	2	12	12	12	1	12	1	12	1	12
12	2	12	12	12	1	12	1	12	1	12	12	12	2	12	12
1	12	1	12	1	12	12	12	2	12	12	12	12	2	12	12
12	12	12	2	12	12	12	12	2	12	12	12	1	12	1	12
12	12	12	2	12	12	12	1	12	1	12	1	12	12	12	2
12	12	1	12	1	12	1	12	12	12	2	12	12	12	12	2
12	1	12	12	12	2	12	12	12	12	2	12	12	12	1	12
2	12	12	12	12	2	12	12	12	1	12	1	12	1	12	12

$d = 14$**:** $\delta_{14} = 28/39$ **(halo).**

14	1	14	1	14	14	14	1	14	2	2	14	1	14	14	14	1	14	1	14	14	14
1	14	14	14	1	14	1	14	14	14	14	2	14	14	14	14	14	1	14	14	14	14
14	14	14	14	14	1	14	14	14	14	14	2	14	14	14	14	1	14	1	14	14	14
14	14	14	14	1	14	1	14	14	14	1	14	2	2	14	1	14	14	14	1	14	1
2	2	14	1	14	14	14	1	14	1	14	14	14	14	2	14	14	14	14	14	1	14
14	14	2	14	14	14	14	14	1	14	14	14	14	14	2	14	14	14	14	1	14	1
14	14	2	14	14	14	14	1	14	1	14	14	14	1	14	2	2	14	1	14	14	14
14	1	14	2	2	14	1	14	14	14	1	14	1	14	14	14	14	2	14	14	14	14
1	14	14	14	14	2	14	14	14	14	14	1	14	14	14	14	14	2	14	14	14	14
14	14	14	14	14	2	14	14	14	14	1	14	1	14	14	14	1	14	2	2	14	1
1	14	14	14	1	14	2	2	14	1	14	14	14	1	14	1	14	14	14	14	2	14
14	1	14	1	14	14	14	14	2	14	14	14	14	14	1	14	14	14	14	14	2	14
14	14	1	14	14	14	14	14	2	14	14	14	14	1	14	1	14	14	14	1	14	2
14	1	14	1	14	14	14	1	14	2	2	14	1	14	14	14	1	14	1	14	14	14
1	14	14	14	1	14	1	14	14	14	14	2	14	14	14	14	14	1	14	14	14	14
14	14	14	14	14	1	14	14	14	14	14	2	14	14	14	14	1	14	1	14	14	14
14	14	14	14	1	14	1	14	14	14	1	14	2	2	14	1	14	14	14	1	14	1
2	2	14	1	14	14	14	1	14	1	14	14	14	14	2	14	14	14	14	14	1	14
14	14	2	14	14	14	14	14	1	14	14	14	14	14	2	14	14	14	14	1	14	1
14	14	2	14	14	14	14	1	14	1	14	14	14	1	14	2	2	14	1	14	14	14
14	1	14	2	2	14	1	14	14	14	1	14	1	14	14	14	14	2	14	14	14	14
1	14	14	14	14	2	14	14	14	14	14	1	14	14	14	14	14	2	14	14	14	14

$d = 16$**:** $\delta_{16} = 8/11$ **(halo).**

16	16	1	16	1	16	1	16	16	16	16	2	16	16	16	1	16	16
16	16	16	2	16	16	16	1	16	16	16	2	16	16	16	16	1	16
16	16	16	2	16	16	16	16	1	16	1	16	1	16	16	16	16	2
1	16	1	16	1	16	16	16	16	2	16	16	16	1	16	16	16	2
16	2	16	16	16	1	16	16	16	2	16	16	16	16	1	16	1	16
16	2	16	16	16	16	1	16	1	16	1	16	16	16	16	2	16	16
1	16	1	16	16	16	16	2	16	16	16	1	16	16	16	2	16	16
16	16	16	1	16	16	16	2	16	16	16	16	1	16	1	16	1	16
16	16	16	16	1	16	1	16	1	16	16	16	16	2	16	16	16	1
1	16	16	16	16	2	16	16	16	1	16	16	16	2	16	16	16	16
16	1	16	16	16	2	16	16	16	16	1	16	1	16	1	16	16	16
16	16	1	16	1	16	1	16	16	16	16	2	16	16	16	1	16	16
16	16	16	2	16	16	16	1	16	16	16	2	16	16	16	16	1	16
16	16	16	2	16	16	16	16	1	16	1	16	1	16	16	16	16	2
1	16	1	16	1	16	16	16	16	2	16	16	16	1	16	16	16	2
16	2	16	16	16	1	16	16	16	2	16	16	16	16	1	16	1	16
16	2	16	16	16	16	1	16	1	16	1	16	16	16	16	2	16	16
1	16	1	16	16	16	16	2	16	16	16	1	16	16	16	2	16	16

$d = 17$: $\delta_{17} = 17/23$ **(halo).**

17	1	17	17	17	17	17	2	17	17	17	1	17	1	17	17	17	1	17	1	17	1	17	17
1	17	1	17	17	17	1	17	1	17	1	17	17	17	1	17	1	17	17	17	1	17	17	17
17	17	17	1	17	1	17	17	17	1	17	17	17	17	17	1	17	17	17	17	2	17	17	17
17	17	17	17	1	17	17	17	17	2	17	17	17	17	1	17	17	17	17	17	2	17	17	17
17	17	17	1	17	17	17	17	17	2	17	17	17	1	17	1	17	17	17	1	17	1	17	1
17	17	1	17	1	17	17	17	1	17	1	17	1	17	17	17	1	17	1	17	17	17	1	17
17	1	17	17	17	1	17	1	17	17	17	1	17	17	17	17	17	1	17	17	17	17	2	17
1	17	17	17	17	17	1	17	17	17	17	2	17	17	17	17	1	17	17	17	17	17	2	17
2	17	17	17	17	1	17	17	17	17	17	2	17	17	17	1	17	1	17	17	17	1	17	1
2	17	17	17	1	17	1	17	17	17	1	17	1	17	1	17	17	17	1	17	1	17	17	17
17	1	17	1	17	17	17	1	17	1	17	17	17	1	17	17	17	17	17	1	17	17	17	17
17	17	1	17	17	17	17	17	1	17	17	17	17	2	17	17	17	17	1	17	17	17	17	17
17	17	2	17	17	17	17	1	17	17	17	17	17	2	17	17	17	1	17	1	17	17	17	1
17	17	2	17	17	17	1	17	1	17	17	17	1	17	1	17	1	17	17	17	1	17	1	17
17	1	17	1	17	1	17	17	17	1	17	1	17	17	17	1	17	17	17	17	17	1	17	17
1	17	17	17	1	17	17	17	17	17	1	17	17	17	17	2	17	17	17	17	1	17	17	17
17	17	17	17	2	17	17	17	17	1	17	17	17	17	17	2	17	17	17	1	17	1	17	17
17	17	17	17	2	17	17	17	1	17	1	17	17	17	1	17	1	17	1	17	17	17	1	17
17	17	17	1	17	1	17	1	17	17	17	1	17	1	17	17	17	1	17	17	17	17	17	1
1	17	1	17	17	17	1	17	17	17	17	17	1	17	17	17	17	2	17	17	17	17	1	17
17	1	17	17	17	17	2	17	17	17	17	1	17	17	17	17	17	2	17	17	17	1	17	1
1	17	17	17	17	17	2	17	17	17	1	17	1	17	17	17	1	17	1	17	1	17	17	17
17	1	17	17	17	1	17	1	17	1	17	17	17	1	17	1	17	17	17	1	17	17	17	17
17	17	1	17	1	17	17	17	1	17	17	17	17	17	1	17	17	17	17	2	17	17	17	17

$d = 19$: $\delta_{19} = 38/51$ (halo).

19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	1	19	1	19
19	19	19	19	2	19	19	19	19	1	19	1	19	19	19	1	19	2	2	19	1	19	19	19	1
19	19	19	1	19	2	2	19	1	19	19	19	1	19	1	19	19	19	19	1	19	19	19	19	19
1	19	1	19	19	19	19	1	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19
19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	1
19	2	19	19	19	19	19	2	19	19	19	19	1	19	1	19	19	19	1	19	2	2	19	1	19
1	19	1	19	19	19	1	19	2	2	19	1	19	19	19	1	19	1	19	19	19	19	1	19	19
19	19	19	1	19	1	19	19	19	19	1	19	19	19	19	19	2	19	19	19	19	19	2	19	19
19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19
19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	1	19	1	19	19	19	1	19	2	2
19	19	19	1	19	1	19	19	19	1	19	2	2	19	1	19	19	19	1	19	1	19	19	19	19
2	19	1	19	19	19	1	19	1	19	19	19	19	1	19	19	19	19	19	2	19	19	19	19	19
19	1	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19
19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	1	19	1	19	19	19	1
19	2	19	19	19	19	1	19	1	19	19	19	1	19	2	2	19	1	19	19	19	1	19	1	19
1	19	2	2	19	1	19	19	19	1	19	1	19	19	19	19	1	19	19	19	19	19	2	19	19
19	19	19	19	1	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19
19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	1	19	1	19
19	19	19	19	2	19	19	19	19	1	19	1	19	19	19	1	19	2	2	19	1	19	19	19	1
19	19	19	1	19	2	2	19	1	19	19	19	1	19	1	19	19	19	19	1	19	19	19	19	19
1	19	1	19	19	19	19	1	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19
19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	19	2	19	19	19	19	1
19	2	19	19	19	19	19	2	19	19	19	19	1	19	1	19	19	19	1	19	2	2	19	1	19
1	19	1	19	19	19	1	19	2	2	19	1	19	19	19	1	19	1	19	19	19	19	1	19	19
19	19	19	1	19	1	19	19	19	19	1	19	19	19	19	19	2	19	19	19	19	19	2	19	19

$d = 22$**:** $\delta_{22} = 22/29$ **(halo).**

2	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22
22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22	22	22	22	1	22	1	22	1	22
22	22	1	22	22	22	1	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1
22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	22	2	22	22	22	22
22	22	22	22	1	22	22	22	1	22	22	22	22	2	22	22	22	22	1	22	22	22	2	22	22	22	22
22	22	22	22	2	22	22	22	22	1	22	22	22	2	22	22	22	22	22	1	22	1	22	1	22	22	22
1	22	22	22	2	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22
22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22	22	22	22	1	22
22	22	1	22	22	22	1	22	22	22	1	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1
22	1	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	22	2
1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	22	2	22	22	22	22	1	22	22	22	2
22	22	22	1	22	22	22	22	2	22	22	22	22	1	22	22	22	2	22	22	22	22	22	1	22	1	22
22	22	22	22	1	22	22	22	2	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22
22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22	22
1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22	22	22	22	1	22	1	22	1	22	22
22	1	22	22	22	1	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22
22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	22	2	22	22	22	22	1
22	22	22	1	22	22	22	1	22	22	22	22	2	22	22	22	22	1	22	22	22	2	22	22	22	22	22
22	22	22	2	22	22	22	22	1	22	22	22	2	22	22	22	22	22	1	22	1	22	1	22	22	22	22
22	22	22	2	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22
1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22	22	22	22	1	22	1
22	1	22	22	22	1	22	22	22	1	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22
1	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	22	2	22
22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	22	2	22	22	22	22	1	22	22	22	2	22
22	22	1	22	22	22	22	2	22	22	22	22	1	22	22	22	2	22	22	22	22	22	1	22	1	22	1
22	22	22	1	22	22	22	2	22	22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22
22	22	22	22	1	22	1	22	1	22	22	22	22	22	1	22	22	22	1	22	22	22	1	22	22	22	22

$d = 23$**:** $\delta_{23} = 23/30$ **(halo).**

23	23	23	23	23	1	23	23	23	1	23	1	23	1	23	23	23	1	23	23
1	23	23	23	1	23	1	23	1	23	23	23	1	23	23	23	23	23	1	23
23	1	23	1	23	23	23	1	23	23	23	23	23	1	23	23	23	23	23	1
23	23	1	23	23	23	23	23	1	23	23	23	23	23	1	23	23	23	23	23
23	23	23	1	23	23	23	23	23	1	23	23	23	23	23	1	23	23	23	1
23	23	23	23	1	23	23	23	23	23	1	23	23	23	1	23	1	23	1	23
23	23	23	23	23	1	23	23	23	1	23	1	23	1	23	23	23	1	23	23
1	23	23	23	1	23	1	23	1	23	23	23	1	23	23	23	23	23	1	23
23	1	23	1	23	23	23	1	23	23	23	23	23	1	23	23	23	23	23	1
23	23	1	23	23	23	23	23	1	23	23	23	23	23	1	23	23	23	23	23
23	23	23	1	23	23	23	23	23	1	23	23	23	23	23	1	23	23	23	1
23	23	23	23	1	23	23	23	23	23	1	23	23	23	1	23	1	23	1	23
23	23	23	23	23	1	23	23	23	1	23	1	23	1	23	23	23	1	23	23
1	23	23	23	1	23	1	23	1	23	23	23	1	23	23	23	23	23	1	23
23	1	23	1	23	23	23	1	23	23	23	23	23	1	23	23	23	23	23	1
23	23	1	23	23	23	23	23	1	23	23	23	23	23	1	23	23	23	23	23
23	23	23	1	23	23	23	23	23	1	23	23	23	23	23	1	23	23	23	1
23	23	23	23	1	23	23	23	23	23	1	23	23	23	1	23	1	23	1	23
23	23	23	23	23	1	23	23	23	1	23	1	23	1	23	23	23	1	23	23
1	23	23	23	1	23	1	23	1	23	23	23	1	23	23	23	23	23	1	23

$d = 24$: $\delta_{24} = 24/31$ **(halo).**

1	24	1	24	24	24	24	24	2	24	24	24	24	24	24	2	24	24	24	24
24	2	24	24	24	24	24	24	2	24	24	24	24	24	1	24	1	24	24	24
24	2	24	24	24	24	24	1	24	1	24	24	24	1	24	24	24	1	24	1
1	24	1	24	24	24	1	24	24	24	1	24	1	24	24	24	24	24	2	24
24	24	24	1	24	1	24	24	24	24	24	2	24	24	24	24	24	24	2	24
24	24	24	24	2	24	24	24	24	24	24	2	24	24	24	24	24	1	24	1
24	24	24	24	2	24	24	24	24	24	1	24	1	24	24	24	1	24	24	24
24	24	24	1	24	1	24	24	24	1	24	24	24	1	24	1	24	24	24	24
24	24	1	24	24	24	1	24	1	24	24	24	24	24	2	24	24	24	24	24
24	1	24	24	24	24	24	2	24	24	24	24	24	24	2	24	24	24	24	24
2	24	24	24	24	24	24	2	24	24	24	24	24	1	24	1	24	24	24	1
2	24	24	24	24	24	1	24	1	24	24	24	1	24	24	24	1	24	1	24
24	1	24	24	24	1	24	24	24	1	24	1	24	24	24	24	24	2	24	24
24	24	1	24	1	24	24	24	24	24	2	24	24	24	24	24	24	2	24	24
24	24	24	2	24	24	24	24	24	24	2	24	24	24	24	24	1	24	1	24
24	24	24	2	24	24	24	24	24	1	24	1	24	24	24	1	24	24	24	1
24	24	1	24	1	24	24	24	1	24	24	24	1	24	1	24	24	24	24	24
24	1	24	24	24	1	24	1	24	24	24	24	24	2	24	24	24	24	24	24
1	24	24	24	24	24	2	24	24	24	24	24	24	2	24	24	24	24	24	1
24	24	24	24	24	24	2	24	24	24	24	24	1	24	1	24	24	24	1	24

$d = 3$**:** $\delta_3 \geq 18/35$ **(improves** $1/2$**; halo bound** $6/11$**).**

3	3	2	2	1	3	3	1	3	2	1	3	3	2	3	1	3	1	3	3	2
1	3	1	3	3	2	2	3	3	2	3	1	3	2	3	3	2	2	3	1	3
3	2	2	3	1	3	3	2	2	1	3	3	1	3	2	1	3	3	2	3	1
1	3	3	2	3	1	3	1	3	3	2	2	3	3	2	3	1	3	2	3	3
3	1	3	2	3	3	2	2	3	1	3	3	2	2	1	3	3	1	3	2	1
3	3	1	3	2	1	3	3	2	3	1	3	1	3	3	2	2	3	3	2	3
2	2	3	3	2	3	1	3	2	3	3	2	2	3	1	3	3	2	2	1	3
3	3	2	2	1	3	3	1	3	2	1	3	3	2	3	1	3	1	3	3	2
1	3	1	3	3	2	2	3	3	2	3	1	3	2	3	3	2	2	3	1	3
3	2	2	3	1	3	3	2	2	1	3	3	1	3	2	1	3	3	2	3	1
1	3	3	2	3	1	3	1	3	3	2	2	3	3	2	3	1	3	2	3	3
3	1	3	2	3	3	2	2	3	1	3	3	2	2	1	3	3	1	3	2	1
3	3	1	3	2	1	3	3	2	3	1	3	1	3	3	2	2	3	3	2	3
2	2	3	3	2	3	1	3	2	3	3	2	2	3	1	3	3	2	2	1	3
3	3	2	2	1	3	3	1	3	2	1	3	3	2	3	1	3	1	3	3	2
1	3	1	3	3	2	2	3	3	2	3	1	3	2	3	3	2	2	3	1	3
3	2	2	3	1	3	3	2	2	1	3	3	1	3	2	1	3	3	2	3	1
1	3	3	2	3	1	3	1	3	3	2	2	3	3	2	3	1	3	2	3	3
3	1	3	2	3	3	2	2	3	1	3	3	2	2	1	3	3	1	3	2	1
3	3	1	3	2	1	3	3	2	3	1	3	1	3	3	2	2	3	3	2	3
2	2	3	3	2	3	1	3	2	3	3	2	2	3	1	3	3	2	2	1	3

$d = 6$: $\delta_6 \geq 18/29$ **(improves** $3/5$**; halo bound** $12/19$**).**

2	6	6	1	6	2	6	6	6	1	6	1	6	1	6	6	6	2	6
2	6	6	6	1	6	1	6	1	6	6	6	2	6	1	6	6	2	6
6	1	6	1	6	6	6	2	6	1	6	6	2	6	6	2	2	6	6
6	6	2	6	1	6	6	2	6	6	2	2	6	6	2	6	6	1	6
6	6	2	6	6	2	2	6	6	2	6	6	1	6	2	6	6	6	1
2	2	6	6	2	6	6	1	6	2	6	6	6	1	6	1	6	1	6
6	6	1	6	2	6	6	6	1	6	1	6	1	6	6	6	2	6	1
6	6	6	1	6	1	6	1	6	6	6	2	6	1	6	6	2	6	6
1	6	1	6	6	6	2	6	1	6	6	2	6	6	2	2	6	6	2
6	2	6	1	6	6	2	6	6	2	2	6	6	2	6	6	1	6	2
6	2	6	6	2	2	6	6	2	6	6	1	6	2	6	6	6	1	6
2	6	6	2	6	6	1	6	2	6	6	6	1	6	1	6	1	6	6
6	1	6	2	6	6	6	1	6	1	6	1	6	6	6	2	6	1	6
6	6	1	6	1	6	1	6	6	6	2	6	1	6	6	2	6	6	2
6	1	6	6	6	2	6	1	6	6	2	6	6	2	2	6	6	2	6
2	6	1	6	6	2	6	6	2	2	6	6	2	6	6	1	6	2	6
2	6	6	2	2	6	6	2	6	6	1	6	2	6	6	6	1	6	1
6	6	2	6	6	1	6	2	6	6	6	1	6	1	6	1	6	6	6
1	6	2	6	6	6	1	6	1	6	1	6	6	6	2	6	1	6	6

$d = 10$: $\delta_{10} \geq 40/59$ (improves $2/3$; halo bound $20/29$).

10	10	10	2	2	10	1	10	10	1	10	10	10	10	2	2	10	10	10	10	1	10	10	1	10	2
10	10	1	10	10	10	10	2	2	10	10	10	10	1	10	10	1	10	2	2	10	10	10	10	2	10
2	2	10	10	10	10	1	10	10	1	10	2	2	10	10	10	10	2	10	10	1	10	10	10	2	10
10	10	1	10	2	2	10	10	10	10	2	10	10	1	10	10	10	2	10	10	10	1	10	1	10	1
10	10	10	2	10	10	1	10	10	10	2	10	10	10	1	10	1	10	1	10	10	10	2	10	10	10
10	10	10	2	10	10	10	1	10	1	10	1	10	10	10	2	10	10	10	1	10	10	2	10	10	10
1	10	1	10	1	10	10	10	2	10	10	10	1	10	10	2	10	10	10	10	2	2	10	1	10	10
10	2	10	10	10	1	10	10	2	10	10	10	10	2	2	10	1	10	10	1	10	10	10	10	2	2
10	2	10	10	10	10	2	2	10	1	10	10	1	10	10	10	10	2	2	10	10	10	10	1	10	10
2	10	1	10	10	1	10	10	10	10	2	2	10	10	10	10	1	10	10	1	10	2	2	10	10	10
10	10	10	2	2	10	10	10	10	1	10	10	1	10	2	2	10	10	10	10	2	10	10	1	10	10
10	10	1	10	10	1	10	2	2	10	10	10	10	2	10	10	1	10	10	10	2	10	10	10	1	10
2	2	10	10	10	10	2	10	10	1	10	10	10	2	10	10	10	1	10	1	10	1	10	10	10	2
10	10	1	10	10	10	2	10	10	10	1	10	1	10	1	10	10	10	2	10	10	10	1	10	10	2
10	10	10	1	10	1	10	1	10	10	10	2	10	10	10	1	10	10	2	10	10	10	10	2	2	10
1	10	10	10	2	10	10	10	1	10	10	2	10	10	10	10	2	2	10	1	10	10	1	10	10	10
10	1	10	10	2	10	10	10	10	2	2	10	1	10	10	1	10	10	10	10	2	2	10	10	10	10
10	10	2	2	10	1	10	10	1	10	10	10	10	2	2	10	10	10	10	1	10	10	1	10	2	2
10	1	10	10	10	10	2	2	10	10	10	10	1	10	10	1	10	2	2	10	10	10	10	2	10	10
2	10	10	10	10	1	10	10	1	10	2	2	10	10	10	10	2	10	10	1	10	10	10	2	10	10
10	1	10	2	2	10	10	10	10	2	10	10	1	10	10	10	2	10	10	10	1	10	1	10	1	10
10	10	2	10	10	1	10	10	10	2	10	10	10	1	10	1	10	1	10	10	10	2	10	10	10	1
10	10	2	10	10	10	1	10	1	10	1	10	10	10	2	10	10	10	1	10	10	2	10	10	10	10
10	1	10	1	10	10	10	2	10	10	10	1	10	10	2	10	10	10	10	2	2	10	1	10	10	1
2	10	10	10	1	10	10	2	10	10	10	10	2	2	10	1	10	10	1	10	10	10	10	2	2	10
2	10	10	10	10	2	2	10	1	10	10	1	10	10	10	10	2	2	10	10	10	10	1	10	10	1

$d = 20$: $\delta_{20} \geq 20/27$ (improves $5/7$; halo bound $40/53$).

1	2	20	20	20	20	20	2	20	20	20	20	20	1	20	20	20	20	1
20	2	20	20	20	20	20	1	20	20	20	20	1	20	1	20	20	1	20
20	1	20	20	20	20	1	20	1	20	20	1	20	20	20	1	2	20	20
1	20	1	20	20	1	20	20	20	1	2	20	20	20	20	20	2	20	20
20	20	20	1	2	20	20	20	20	20	2	20	20	20	20	20	1	20	20
20	20	20	20	2	20	20	20	20	20	1	20	20	20	20	1	20	1	20
20	20	20	20	1	20	20	20	20	1	20	1	20	20	1	20	20	20	1
20	20	20	1	20	1	20	20	1	20	20	20	1	2	20	20	20	20	20
20	20	1	20	20	20	1	2	20	20	20	20	20	2	20	20	20	20	20
1	2	20	20	20	20	20	2	20	20	20	20	20	1	20	20	20	20	1
20	2	20	20	20	20	20	1	20	20	20	20	1	20	1	20	20	1	20
20	1	20	20	20	20	1	20	1	20	20	1	20	20	20	1	2	20	20
1	20	1	20	20	1	20	20	20	1	2	20	20	20	20	20	2	20	20
20	20	20	1	2	20	20	20	20	20	2	20	20	20	20	20	1	20	20
20	20	20	20	2	20	20	20	20	20	1	20	20	20	20	1	20	1	20
20	20	20	20	1	20	20	20	20	1	20	1	20	20	1	20	20	20	1
20	20	20	1	20	1	20	20	1	20	20	20	1	2	20	20	20	20	20
20	20	1	20	20	20	1	2	20	20	20	20	20	2	20	20	20	20	20
1	2	20	20	20	20	20	2	20	20	20	20	20	1	20	20	20	20	1

4 Summary

Table 2 lists the resulting bounds. A bound or a closure is shown in **green** when it is new here; entries already in the book are in plain type. In all, δ_d is now determined for 19 of the 25 values, the open cases being $d \in \{3, 6, 10, 15, 20, 21\}$.

d	lower bound	upper bound	determined?
1	3/8	**3/8**	**yes**
2	4/9	**4/9**	**yes**
3	**18/35**	6/11	open
4	**4/7**	4/7	**yes**
5	5/8	5/8	yes
6	**18/29**	12/19	open
7	**7/11**	7/11	**yes**
8	2/3	2/3	yes
9	**2/3**	2/3	**yes**
10	**40/59**	20/29	open
11	11/16	11/16	yes
12	**12/17**	12/17	**yes**
13	13/18	13/18	yes
14	**28/39**	28/39	**yes**
15	5/7	30/41	open
16	**8/11**	8/11	**yes**
17	**17/23**	17/23	**yes**
18	3/4	3/4	yes
19	**38/51**	38/51	**yes**
20	**20/27**	40/53	open
21	3/4	42/55	open
22	**22/29**	22/29	**yes**
23	**23/30**	23/30	**yes**
24	**24/31**	24/31	**yes**
25	25/32	25/32	yes

Table 2: Bounds after this note. **Green** marks a new lower bound, a new upper bound, or a new closure; plain entries are from the book. For $d = 1, 2$ the upper bound is the combinatorial one of Section 3.1; the remaining upper bounds are the halo bound, met by the construction of Section 3.2 at the green lower bounds.