Stappers' construction yields *all* maximum 3-uncolourability preserving edge sets of the pinched Sierpiński gasket graph

(a proof, for all n, of the conjecture in Answer 7.2.2.3–227 of *TAOCP*)

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June 2026

Abstract

The Sierpiński gasket graph $S_n^{(3)}$ (the d=3 Sierpiński simplex graph) has three corner vertices $0\cdots 0$, $1\cdots 1$, $2\cdots 2$; identifying the first two yields the *pinched* graph $\widehat{S}_n^{(3)}$, which is not 3-colourable. Call a set of edges *removable* if deleting it leaves the graph still not 3-colourable. In the Answer to exercise 7.2.2.3–227 of *The Art of Computer Programming*, D. E. Knuth records a construction of Filip Stappers that produces 2^{n-1} removable sets of $2^{n-1}+1$ edges each, and asks whether this construction actually produces *all* of the largest removable sets for every n (it was verified only for $n \le 4$). We prove that it does. The proof reduces the question, through the gasket self-similarity of $S_n^{(3)}$, to a finite transfer system on the 15 possible "realizable corner-colour sets"; this system is closed under the gluing composition, and its cost-optimal decompositions are eventually constant, so a short induction gives that the maximum removable sets number exactly 2^{n-1} and have $2^{n-1}+1$ edges for every $n \ge 3$ (the range $n > 2$ of Stappers' construction). As there are exactly that many, Stappers' (equally many, distinct) sets are all of them.

1 Introduction

For $d \ge 2$ and $n \ge 1$, the *Sierpiński simplex graph* $S_n^{(d)}$ is obtained from the strings $a_1 \cdots a_n$ with $0 \le a_j < d$ by joining $a_1 \cdots a_{n-1} j$ to $a_1 \cdots a_{n-1} k$ for $j \ne k$ (the *clique* edges) and by *identifying*

$$a_1 \cdots a_i \, j \, k \cdots k \;\equiv\; a_1 \cdots a_i \, k \, j \cdots j \qquad (0 \le i < n-1,\ j \ne k), \tag{1}$$

so that each "non-clique" edge of the proper Sierpiński graph collapses to a single vertex. We work throughout with $d = 3$ and write $S_n := S_n^{(3)}$; for $d = 3$ this is the *Sierpiński gasket graph*, and its pinched form $\widehat{S}_n$ is the *pinched Sierpiński gasket* of exercise 227. The graph S_n has three *corner* vertices,

$$A = 0^n, \qquad B = 1^n, \qquad C = 2^n,$$

and $|E(S_n)| = 3^n$ edges (the 3^{n-1} clique-prefixes of length $n-1$ each carry $\binom{3}{2} = 3$ edges). The *pinched* graph $\widehat{S}_n$ is S_n with A and B identified; it is not 3-colourable (Proposition 1). *For $n \geq 3$ this identification deletes no edge*: $A = 0^n$ and $B = 1^n$ have no common neighbour (their neighbours are $0^{n-1}1, 0^{n-1}2$ and $1^{n-1}0, 1^{n-1}2$, pairwise distinct as vertices once $n \geq 3$), so $E(\widehat{S}_n) = E(S_n)$ as edge sets and a removable set of $\widehat{S}_n$ is literally a removed set of S_n. (At $n = 2$, by contrast, $A = 00$ and $B = 11$ *do* share the neighbour $g_{AB} = 01$, so the two edges $A-01$ and $B-01$ coincide once A, B are identified — pinching here *deletes* an edge. The maximum is then $\mu_2 = 2$ (only the two tip edges) and $\nu_2 = 1$: genuinely different, and outside Stappers' range $n > 2$. We assume $n \geq 3$ throughout.)

Definition 1. A set $S \subseteq E(\widehat{S}_n)$ is *removable* if $\widehat{S}_n - S$ is still not 3-colourable. Let μ_n be the maximum size of a removable set and ν_n the number of removable sets of size μ_n.

In the Answer to exercise 7.2.2.3–227, Knuth gives Stappers' construction of 2^{n-1} removable sets, each of size $2^{n-1}+1$, every one containing the two "tip" edges at the corner $2 \cdots 2$; he remarks that for $n \leq 4$ this produces *all* of the largest removable sets, and asks: *is that conjecture actually true for all n?* Our main result answers this affirmatively.

Theorem 1. *For every $n \geq 3$,*

$$\mu_n = 2^{n-1} + 1, \qquad \nu_n = 2^{n-1}.$$

Consequently Stappers' construction — which produces 2^{n-1} distinct removable sets of size $2^{n-1}+1$ — produces all *of the maximum removable sets, for every $n \geq 3$.*

The last sentence is immediate from the count: a family of 2^{n-1} distinct removable sets, each of the maximum size $2^{n-1}+1$, sits inside the set of *all* maximum removable sets, which by Theorem 1 also has exactly 2^{n-1} members; so the two families coincide.

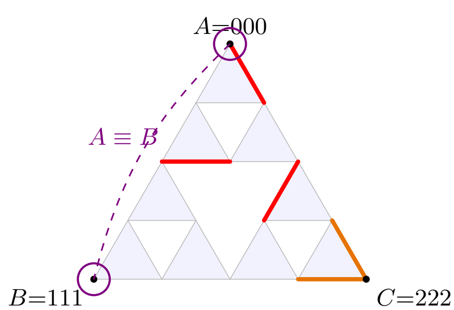

Figure 1: A maximum removable set of $\widehat{S}_3$ (the case base=0, d_3=0 of Stappers' construction): the two tip edges at corner C=222 (present in every set) together with three non-tip edges, $5 = 2^2+1$ edges in all. The corners A=000 and B=111 are pinched ($A \equiv B$). Deleting these five edges leaves $\widehat{S}_3$ still not 3-colourable; there are $\nu_3 = 4$ such sets.

2 The gasket recursion and a colouring reformulation

Corners are always distinct. Throughout, a 3-*colouring* of a graph is a proper one (adjacent vertices get different colours), with colours in $\{0, 1, 2\}$.

Proposition 1. *In every* 3*-colouring of* S_n *the three corners* A, B, C *receive distinct colours. In particular* $\widehat{S}_n$ *is not* 3*-colourable, and*

$$S \textit{ is removable} \iff \textit{every } 3\textit{-colouring of } S_n - S \textit{ gives } A, B \textit{ different colours.} \tag{2}$$

Proof. We show, by induction on n, that $\rho(S_n) = D$ (no edges removed), where $\rho(\cdot)$ and the composition law (4) are introduced just below and do *not* depend on this Proposition. The base $S_1 = K_3$ has $\rho = D$. For the step, the three copies $C_0, C_1, C_2 \cong S_{n-1}$ have $\rho(C_i) = D$ by the induction hypothesis, so by (4) $\rho(S_n) = \mathrm{COMPOSE}(D, D, D)$. A direct check gives $\mathrm{COMPOSE}(D, D, D) = D$: if $(a, p, q), (p, b, r), (q, r, c) \in D$ then a, p, q distinct and p, b, r distinct force, when $a = b$, that q and r are both the unique colour $\neq a, p$, so $q = r$, whence $(q, r, c) = (q, q, c) \notin D$ — contradiction; thus $a \neq b$. The gasket carries a 3-fold rotational symmetry cyclically permuting (A, B, C) and the copies (C_0, C_1, C_2), under which the same argument gives $b \neq c$ and $c \neq a$; hence all of a, b, c are distinct, i.e. $\mathrm{COMPOSE}(D, D, D) \subseteq D$ (and $\supseteq D$ trivially; the equality is also a one-line finite check). Hence $\rho(S_n) = D$: the three corners are always distinctly coloured.

(The naive induction "each copy has distinct corners, therefore S_n does" is *insufficient*: $A \neq g_{AB}$ and $B \neq g_{AB}$ do not force $A \neq B$. The computation $\mathrm{COMPOSE}(D, D, D) = D$ is exactly what supplies the missing step.) □

A 3-colouring of $\widehat{S}_n - S$ is the same thing as a 3-colouring of $S_n - S$ in which A and B receive equal colours (no edge is lost when $n \geq 3$); hence (2).

Three glued copies. The map $a_1 \cdots a_n \mapsto a_1$ splits S_n into three induced copies C_0, C_1, C_2 of S_{n-1}, where C_i consists of the strings beginning with i. Because every clique edge has a well-defined first symbol, *every edge of* S_n *lies in exactly one copy*; the copies meet only at three shared corner vertices, which by (1) are

$$g_{AB} = 0\,1^{n-1} = 1\,0^{n-1}, \quad g_{AC} = 0\,2^{n-1} = 2\,0^{n-1}, \quad g_{BC} = 1\,2^{n-1} = 2\,1^{n-1}.$$

Listing each copy's corners in the order (suffix 0^{n-1}, 1^{n-1}, 2^{n-1}), we have

$$C_0 : (A, g_{AB}, g_{AC}), \qquad C_1 : (g_{AB}, B, g_{BC}), \qquad C_2 : (g_{AC}, g_{BC}, C). \tag{3}$$

Thus a removable set S partitions uniquely as $S = S_0 \sqcup S_1 \sqcup S_2$ with $S_i \subseteq E(C_i)$, and $|S| = |S_0| + |S_1| + |S_2|$.

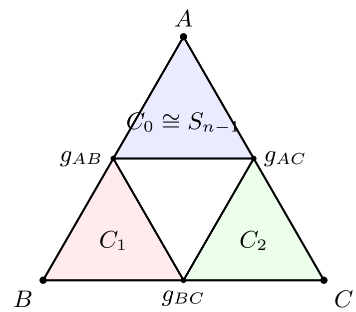

Figure 2: The gasket recursion: S_n is three copies C_0, C_1, C_2 of S_{n-1} meeting only at the three glue corners g_{AB}, g_{AC}, g_{BC} (eq. (3)); the free corners A, B, C are the corners of S_n. Every edge lies in exactly one copy, so a removed set splits uniquely as $S = S_0 \sqcup S_1 \sqcup S_2$.

Realizable corner-colour sets. For a graph H with an ordered triple of distinguished corners, let

$$\rho(H) \;:=\; \big\{\, \text{(colours of the three corners)} \;:\; \text{over all 3-colourings of } H \,\big\} \;\subseteq\; \{0,1,2\}^3.$$

Write $D := \{(x, y, z) : x, y, z \text{ distinct}\}$; by Proposition 1, $\rho(S_m) = D$, and for any removed set $\rho \supseteq D$ (deleting edges only adds colourings). A 3-colouring of $S_n - S$ assigns colours a, b, c to A, B, C and p, q, r to g_{AB}, g_{AC}, g_{BC}; it restricts to a colouring of each copy, and conversely three copy-colourings glue iff they agree on shared corners. With (3) this gives the *composition law*

$$\rho(S_n - S) = \mathrm{COMPOSE}(\rho_0, \rho_1, \rho_2) := \Big\{(a, b, c) : \exists\, p, q, r,\ (a, p, q) \in \rho_0,\ (p, b, r) \in \rho_1,\ (q, r, c) \in \rho_2\Big\}, \tag{4}$$

where $\rho_i := \rho(C_i - S_i)$. Combining (2) and (4): *S is removable iff* $\mathrm{COMPOSE}(\rho_0, \rho_1, \rho_2)$ *is "AB-clean", i.e. contains no triple (a, a, c).*

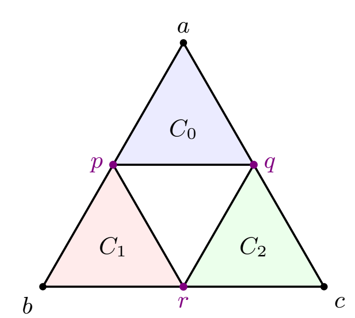

Figure 3: The composition law (4). Each glue corner $p = g_{AB}$, $q = g_{AC}$, $r = g_{BC}$ (violet) is shared by two copies and must get one colour in both; the free corners carry the colours a, b, c of A, B, C. COMPOSE glues by agreeing on p, q, r and projects to (a, b, c); S is removable iff no glued colouring achieves $a = b$.

3 A finite transfer system

Let Σ be the set of *corner spectra* $\rho(S_m - S')$ realizable for some $m \geq 2$; its elements we call *states*. Each such ρ is invariant under global colour permutations — a permutation of $\{0, 1, 2\}$ sends proper colourings to proper colourings — hence is a union of D (all distinct) with some of the colour-orbits

$$E_{01} = \{(a, a, c) : a \neq c\}, \quad E_{02} = \{(a, c, a) : a \neq c\}, \quad E_{12} = \{(c, a, a) : a \neq c\}, \quad T = \{(a, a, a)\},$$

D being always present ($|E_{ij}| = 6$, $|T| = 3$). Every state has the form $\rho = D \cup \bigcup_{ij \in I} E_{ij}$, with T either adjoined or not, for some $I \subseteq \{01, 02, 12\}$: that is, 2^3 choices of I times 2 (whether or not T is included), giving 16 formal possibilities. Exactly one, $D \cup T$ ($|\rho| = 9$), is *not* realizable, leaving the 15 states of Lemma 1. A state is AB-clean iff it contains *neither* E_{01} *nor* T (each makes corners $0, 1$ coincide); the four AB-clean states are D, $D \cup E_{02}$, $D \cup E_{12}$, and $\sigma^\star := D \cup E_{02} \cup E_{12}$.

Lemma 1 (Closure)**.** *Σ has exactly* 15 *elements and is closed under* COMPOSE*: for all $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$ one has* $\mathrm{COMPOSE}(\sigma_0, \sigma_1, \sigma_2) \in \Sigma$.

Proof. Every realizable $\rho(S_m - S')$ is, by (4), a COMPOSE-iterate of the realizable sets of the triangle $S_1 = K_3$, of which there are $2^3 = 8$ (one per subset of its three edges). Closing those 8 under COMPOSE reaches a set of 15 states and no more (each already realized at $m = 2$); checking all $15^3 = 3375$ compositions — a finite verification — confirms closure. □

Up to the symmetry $\mathfrak{S}_3$ that permutes the three corner coordinates, the 15 states fall into the seven orbits of Table 1, indexed by $|\rho|$ and their $D/E/T$ coordinates (a state is AB-clean iff it contains neither E_{01} nor T).

Definition 2. For a state σ and $m \geq 2$ let

$$M_m(\sigma) := \max\{|S'| : \rho(S_m - S') = \sigma\}, \qquad N_m(\sigma) := \#\{S' : \rho(S_m - S') = \sigma,\ |S'| = M_m(\sigma)\}.$$

Because the three copies are edge-disjoint and removed independently, and ρ composes by (4), these obey the recursion

$$M_m(\sigma) = \max_{\substack{(\sigma_0,\sigma_1,\sigma_2) \\ \mathrm{COMPOSE}(\sigma_0,\sigma_1,\sigma_2)=\sigma}} \big(M_{m-1}(\sigma_0) + M_{m-1}(\sigma_1) + M_{m-1}(\sigma_2)\big), \tag{5}$$

$$N_m(\sigma) = \sum_{\text{maximising } (\sigma_0,\sigma_1,\sigma_2)} N_{m-1}(\sigma_0)\, N_{m-1}(\sigma_1)\, N_{m-1}(\sigma_2). \tag{6}$$

(No removed set is counted twice: S' determines each S_i, hence each $\rho_i = \sigma_i$, hence the triple. Moreover, whether a triple $(\sigma_0, \sigma_1, \sigma_2)$ composes to σ depends only on the states σ_i, not on which removed sets S_i realise them; so along a fixed maximising triple the three copies are chosen independently and the counts multiply, as written.) Here the σ_i range over Σ and (5)–(6) hold for $m \geq 3$; the values at $m = 2$ (computed once from the eight triangle states $\rho(K_3 - S')$, all of which lie in Σ) serve as the base of the induction. Every state of Σ is realized at every $m \geq 2$, so M_m, N_m are everywhere defined.

Lemma 2 (Closed forms, for all m)**.** *The entries $M_m(\sigma)$ of Table 1, together with the counts N_m of the AB-clean states displayed explicitly in Table 2, hold for every $m \geq 2$.*

Proof. The system (5)–(6) is a fixed (max-plus, then counting) map on the vector $(M_m(\sigma))_{\sigma\in\Sigma}$ whose support does not depend on m. We claim each $M_m(\sigma)$ has the affine-in-2^m form $M_m(\sigma) = c_\sigma 2^m + e_\sigma$ of Table 1 (with 3^m for the all-triples state); for the AB-clean states it is realised by a maximising triple that is *constant in m*.

Substitute the candidate forms into the right-hand side of (5). Since each $M_{m-1}(\sigma_i) = c_{\sigma_i}2^{m-1} + e_{\sigma_i}$, a triple composing to σ contributes $\big(c_{\sigma_0}+c_{\sigma_1}+c_{\sigma_2}\big)2^{m-1} + \big(e_{\sigma_0}+e_{\sigma_1}+e_{\sigma_2}\big)$ with m-independent $(\sum c, \sum e)$, so the right-hand side is a maximum of finitely many affine functions of 2^{m-1}. (The all-triples state cannot interfere: it is *absorbing* under COMPOSE — any all-triples input forces an all-triples output — so it is reached only by (all, all, all), giving 3^m, and enters no other state's right-hand side; thus no 3^{m-1} term appears for the others.) Such a maximum is, in general, *piecewise* in m: a triple with larger $\sum c$ overtakes one with larger $\sum e$ past some bounded m. What the result needs, however, holds with no threshold — *for each AB-clean state the maximiser is the same for all $m \geq 2$.* This is a finite check (the winning triple's $(\sum c, \sum e)$ already dominates every competitor at $m = 2$, hence at all larger m), and it makes the closed forms M_m and counts N_m of Table 2 valid for every $m \geq 2$, with (6) propagating the counts along these constant maximisers. (A few non-AB-clean states do shift their maximiser at small m — e.g. the 24-state has $N_m = 2, 8, 2, 2, \ldots$ — but they never enter μ_n, ν_n.)

For the four AB-clean states the computation is fully explicit in Table 2. In particular the unique AB-clean state of largest cost, $\sigma^\star$ has the constant maximiser $(D, D\cup E_{01}, \sigma^\star)$ — the all-distinct state in one corner copy, a one-pair state $D\cup E_{01}$ in another, and a recursive $\sigma^\star$ in C_2 — so that

$$M_m(\sigma^\star) = \underbrace{0}_{D} + \underbrace{2^{m-2}}_{D\cup E_{01}} + \underbrace{(2^{m-2}+1)}_{\sigma^\star} = 2^{m-1}+1, \qquad N_m(\sigma^\star) = 2\,N_{m-1}(\sigma^\star) = 2^{m-1},$$

the factor 2 being the binary choice of which of the two corner copies C_0, C_1 carries $D\cup E_{01}$ (the other carrying D). The remaining AB-clean states have strictly smaller cost, so the AB-clean maximum is attained uniquely at $\sigma^\star$. $\square$

4 Proof of Theorem 1

By (2) and the composition law, a removable set of $\widehat{S}_n$ is exactly a removed set S of S_n for which $\rho(S_n - S)$ is AB-clean. Therefore

$$\mu_n = \max_{\sigma\in\Sigma,\ \sigma\ AB\text{-clean}} M_n(\sigma).$$

state σ (orbit rep.)	$\lvert\rho\rvert$	$M_m(\sigma)$	$N_m(\sigma)$	a maximising triple $(\sigma_0, \sigma_1, \sigma_2)$
D	6	0	1	(D, D, D)
$D{\cup}E_{01}$	12	2^{m-1}	1	$(D{\cup}E_{02},\ D{\cup}E_{12},\ D)$
$D{\cup}E_{01}{\cup}T$	15	2^{m-2}	1	$(D,\ D,\ D{\cup}E_{01})$
$D{\cup}E_{01}{\cup}E_{02}$	18	$2^{m-1}{+}1$	2^{m-1}	$(D{\cup}E_{01}{\cup}E_{02},\ D,\ D{\cup}E_{12})$
$D{\cup}E_{01}{\cup}E_{02}{\cup}T$	21	$2^{m-1}{+}2$	$\binom{2^{m-1}}{2}$	$(D,\ D{\cup}E_{02},\ D{\cup}E_{01}{\cup}E_{12}{\cup}T)$
$D{\cup}E_{01}{\cup}E_{02}{\cup}E_{12}$	24	$3{\cdot}2^{m-2}$	$\dagger$	$(D{\cup}E_{01},\ D{\cup}E_{12},\ D{\cup}E_{02})$
$D{\cup}E_{01}{\cup}E_{02}{\cup}E_{12}{\cup}T$	27	3^m	1	(all, all, all)

Table 1: Complete certificate for the 15 states, one $\mathfrak{S}_3$-orbit representative per row (orbit sizes $1, 3, 3, 3, 3, 1, 1$ sum to 15). Every row is checked to satisfy (5)–(6) for all $m \geq 2$: the listed triple $(\sigma_0, \sigma_1, \sigma_2)$ composes to σ and realises M_m, i.e. $\sum_i M_{m-1}(\sigma_i) = M_m(\sigma)$, and it is constant in m. The AB-clean states are exactly those containing neither E_{01} nor T — namely D, $D{\cup}E_{02}$, $D{\cup}E_{12}$ and $\sigma^\star{=}D{\cup}E_{02}{\cup}E_{12}$ (the four of Table 2) — and among them $\sigma^\star$ uniquely maximises M_m, giving $\mu_n = 2^{n-1}{+}1$, $\nu_n = 2^{n-1}$. †The 24-state's count is $N_m = 2, 8, 2, 2, \ldots$ for $m = 2, 3, 4, \ldots$; being non-AB-clean it never enters μ_n, ν_n.

state σ	$\lvert\rho\rvert$	$M_m(\sigma)$	$N_m(\sigma)$	maximising triple
D (all distinct)	6	0	1	(D, D, D)
$D{\cup}E_{02}$	12	2^{m-1}	1	$(D{\cup}E_{01}, D, D{\cup}E_{12})$
$D{\cup}E_{12}$	12	2^{m-1}	1	$(D, D{\cup}E_{01}, D{\cup}E_{02})$
$\sigma^\star{=}D{\cup}E_{02}{\cup}E_{12}$	18	$2^{m-1}+1$	2^{m-1}	$(D, D{\cup}E_{01}, \sigma^\star)$

Table 2: The four AB-clean states and the constant maximising triple realising M_m in (5), valid for all $m \geq 2$. Triples are ordered as $(\sigma_0, \sigma_1, \sigma_2)$ for the copies C_0, C_1, C_2 and each is verified as a fixed ray of (5); $\sigma^\star$ has the *second* maximiser $(D{\cup}E_{01}, D, \sigma^\star)$ (the $C_0 {\leftrightarrow} C_1$ swap), the source of the factor 2 in $N_m(\sigma^\star)$. The building block $D \cup E_{01}$ is itself *not* AB-clean, with $M_m(D \cup E_{01}) = 2^{m-1}$ and $N_m(D \cup E_{01}) = 1$. The maximum over AB-clean states is attained uniquely at $\sigma^\star$, giving $\mu_n = M_n(\sigma^\star) = 2^{n-1} + 1$ and $\nu_n = N_n(\sigma^\star) = 2^{n-1}$.

From Table 2 the AB-clean states are: D (cost 0); the two one-pair states $D \cup E_{02}, D \cup E_{12}$ (cost 2^{n-1}); and the two-pair state $\sigma^\star = D{\cup}E_{02}{\cup}E_{12}$ (cost $2^{n-1}{+}1$). The maximum is attained *uniquely* at $\sigma^\star$, so

$$\mu_n = 2^{n-1} + 1.$$

Since $2^{n-1} + 1 > 2^{n-1}$, every maximum removable set has $\rho(S_n - S) = \sigma^\star$; hence

$$\nu_n = N_n(\sigma^\star) = 2^{n-1},$$

the value in Table 2. This proves Theorem 1; the final "all of them" statement was deduced from the count in the Introduction. □

The optimal decomposition is Stappers'. The constant maximising triple for $\sigma^\star$ in (5) is

$$\big(\underbrace{D}_{0},\ \underbrace{\text{one-pair state}}_{2^{n-2}},\ \underbrace{\sigma^\star}_{2^{n-2}+1}\big),$$

i.e. a maximum removable set of $\widehat{S}_n$ consists of (i) a maximum removable set of the copy C_2 — which, by (3), is itself a copy of $\widehat{S}_{n-1}$ pinched at g_{AC}, g_{BC} and carries $2^{n-2}+1$ edges, including recursively the two tip edges at $2\cdots2$; (ii) 2^{n-2} further edges in *exactly one* of the two corner copies C_0, C_1; and (iii) no edges in the other corner copy. The factor 2 between ν_{n-1} and ν_n is precisely the binary choice of *which* corner copy is used. This recursive "two-fold-per-level, deepest tip at $2\cdots2$" structure is exactly Stappers' "different progeny" construction; for instance at $n=5$ the 16 maximum sets factor as a product of four independent binary choices, one of which is the tip-adjacent pair $\{2\cdots200{-}2\cdots201,\ 2\cdots211{-}2\cdots210\}$ named in Knuth's Answer.

Remark 1. Two scope notes. (i) Everything here is specific to $d=3$; the 15-state system and the value $2^{n-1}+1$ are particular to three colours, and the analogous question for $S_n^{(d)}$ with $d>3$ — is the maximum still $\Theta(2^n)$, governed by an analogous finite state system over $\{0,\dots,d-1\}^3$? — is left open. (ii) The deduction "$\nu_n = 2^{n-1}$, therefore Stappers' family is all of them" uses the fact, stated in Knuth's Answer, that the construction yields 2^{n-1} *distinct* removable sets; our Theorem 1 (the count) is what was missing. The explicit product structure of §4 also re-derives this distinctness directly for $n \le 6$.

5 Small cases and the explicit Stappers form

Unwinding the recurrences (5)–(6) over the 15 states — a finite computation, logically independent of the closed-form argument above — gives, for small n, the values

n	$\lvert V(\widehat{S}_n)\rvert$	$\lvert E\rvert$	μ_n	ν_n
3	14	27	5	4
4	41	81	9	8
5	122	243	17	16
6	365	729	33	32

in agreement with $\mu_n = 2^{n-1}+1$ and $\nu_n = 2^{n-1}$ of Theorem 1 (and with the exceptional values $\mu_2 = 2, \nu_2 = 1$).

Finally, Stappers' recursion can be made fully explicit in clique form $(p,\{a,b\})$ (clique-prefix p, joined symbols a,b): the rule "replace a–b by two edges" reads $(p,\{a,b\}) \mapsto \{(p{\cdot}a,\{a,c\}),(p{\cdot}b,\{b,c\})\}$ with c the third symbol; the added edge is $(2^{n-2}d,\{0,1\})$ for $d\in\{0,1\}$, and the two tips are $(2^{n-1},\{0,2\}),(2^{n-1},\{1,2\})$. This form reproduces Knuth's explicit $n=4$ example, the count 2^{n-1}, and the named tip-variant edge; the resulting sets are pairwise distinct and of size $2^{n-1}+1$, and realise the 2^{n-1} maximum removable sets of Theorem 1.

We stress that Theorem 1 does *not* rely on this explicit form, nor on any external property of the construction: the theorem furnishes the count $\nu_n = 2^{n-1}$, and *any* family of 2^{n-1} distinct removable maximum sets — Stappers' or another — must then equal the whole set of maximum removable sets.