The maximum number of solutions of a Dominosa array (exercise 7.2.2.1–73 of Volume 4B)

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1 The question

The references are to Volume 4B, second printing (November 2022), with the changes in the files `err4b.ps.gz` and `all4b.ps.gz` on the Volume 4B errata page, as downloaded on 18 November 2026. Exercise 7.2.2.1–70 on page 131 defines the game:

> **70.** [*21*] *Dominosa* is a solitaire game in which you "shuffle" the 28 pieces $\boxed{0|0}$, $\boxed{0|1}$, ..., $\boxed{6|6}$ of double-six dominoes and place them at random into a 7×8 frame. Then you write down the number of spots in each cell, put the dominoes away, and try to reconstruct their positions based only on that 7×8 array of numbers. ...

(The book draws the pieces with spots.) Exercise 7.2.2.1–73 on the same page asks

> **73.** [*46*] What's the maximum number of solutions to an instance of Dominosa?

The printed answer (pages 431–432) gives Michael Keller's array with 730,924 solutions. The errata of 10 August 2023 replace it by

> **73.** From 06151146/66611014/56132144/55326224/53632022/34300525/33040054 we get 807,752 solutions, which is the current record. This array, found in 2023 by Germán González-Morris based on previous work by Michael Keller, has the surprising property that *every* candidate placement occurs in at least 35,082 solutions!

Here, as in the book, an array is written row by row, a slash separating the rows.

2 The answer

Theorem (computer-assisted). *The maximum number of solutions of a Dominosa array is* 853,985. *It is attained, for example, by the array*

$$A^* = 00060465/20504626/52012466/45122236/51312333/11143035/61444355,$$

and no array has more solutions.

That A^* has 853,985 solutions is a direct count: A short program that runs through the tilings of the board and tests each of them, or a simple backtracking search, finds them in a fraction of a second. In A^*, as in González-Morris's array, every one of the 97 candidate placements (pairs of adjacent cells, $7 \cdot 7 + 6 \cdot 8 = 97$ of them) occurs in some solution, and in fact in at least 37,207

of them. Many other arrays reach 853,985; I found 19,601 of them that are essentially different, in the sense that no two of them are related by a symmetry of the rectangle and a renaming of the numbers, and I did not try to count them all.

The proof that no array does better is computer-assisted, but its idea is simple, and it can be seen at work on smaller boards: For $n = 2$ it can be followed almost completely by hand, and the next cases can be checked by much smaller independent computations. So these notes take a detour. Section 3 sets up Dominosa with fewer pieces. Section 4 explains the one observation on which everything rests. Sections 5–9 use it to find the maximum for the smaller sets, from the 3×4 board, where all but one step of the argument can be seen in one picture, up to the 6×7 board. Section 10 does the same for the real game, and Section 12 collects the numbers.

3 Dominosa with fewer pieces

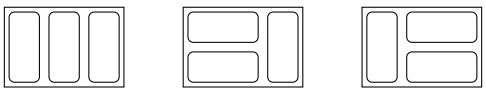
Nothing in the game depends on the number six. The pieces $\{a, b\}$ with $0 \le a \le b \le n$ form a *double-n* set; there are $(n + 1)(n + 2)/2$ of them, and since each piece covers two cells, they exactly fill an $(n+1) \times (n+2)$ board. The double-six game of exercise 70 is the case $n = 6$, with 28 pieces on a 7×8 board. It helps to separate two things that the game mixes together: the empty board, which only knows where dominoes can lie, and the numbers written on it.

3.1 Tilings of the empty board

A *tiling* is a way to cover the empty $(n+1) \times (n+2)$ board with dominoes, each domino covering two cells that are next to each other in a row or in a column. A tiling is just a pattern of dominoes; no numbers are involved. We write

$$T_n = \text{the number of tilings of the empty } (n + 1) \times (n + 2) \text{ board.}$$

It depends on the size of the board and on nothing else. Every tiling counts separately, even when it is the mirror image of another one. For instance, the 2×3 board has exactly three tilings,



and the last two, which are mirror images of each other, count as two. The values of T_n for the boards we shall need are

n	1	2	3	4	5	6
board	2×3	3×4	4×5	5×6	6×7	7×8
T_n	3	11	95	1,183	31,529	1,292,697

The last number appears in answer 72 as the number of "packings" of the 7×8 board.

3.2 Arrays and their solutions

Here and below a piece $\{a, b\}$ is an unordered pair in which $a = b$ is allowed, so that $\{2, 2\}$ is the double two. An *array* is the board with a number from $\{0, 1, \ldots, n\}$ written in each cell; this is what the player of Dominosa is given. Now put an array and a tiling together. Each domino of the tiling covers two cells and therefore two numbers, so it determines a pair, such as $\{0, 3\}$ or $\{2, 2\}$. The tiling is a *solution* of the array if the pairs of its dominoes are all different. Since there are exactly as many dominoes as pieces, the pairs are then all the pieces of the set, each used once: The tiling shows how the pieces could have been placed to produce the array. (The

numbers also decide which way round each piece lies, so nothing besides the tiling needs to be chosen.)

Thus a solution of an array is a tiling with a property that depends on the numbers, and the solutions of an array are some of the T_n tilings of the board. For an array A we write

$$s(A) = \text{the number of tilings that are solutions of } A,$$
$$M_n = \text{the largest value of } s(A) \text{ over all arrays } A.$$

Exercise 73 asks for M_6. Since every solution is one of the tilings of the board, $s(A) \le T_n$ for every array, and so $M_n \le T_n$. The difference between the two numbers is the subject of these notes: T_n counts patterns of dominoes on the empty board, while M_n asks how many of these patterns a single choice of numbers can turn into solutions at once.

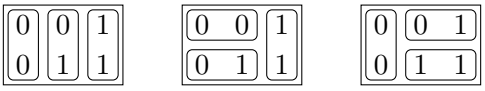
If an array has a solution, every number occurs in it exactly $n+2$ times, because the number k occurs twice on the piece $\{k,k\}$ and once on each of the n other pieces that contain it. So only arrays with $n+2$ copies of each number matter. There are

$$\binom{(n+1)(n+2)}{n+2,\ n+2,\ \ldots,\ n+2} = \frac{\big((n+1)(n+2)\big)!}{\big((n+2)!\big)^{n+1}}$$

of them, because such an array is obtained by choosing which $n+2$ of the $(n+1)(n+2)$ cells receive the number 0, which $n+2$ of the remaining cells receive 1, and so on. For $n=2$ this is $\binom{12}{4}\binom{8}{4} = 495 \cdot 70 = 12!/(4!)^3 = 34{,}650$; for $n=3$ it is $20!/(5!)^4 = 11{,}732{,}745{,}024$; for $n=4$ it is about 1.4×10^{18}; and for the real game, $n=6$, it is $56!/(8!)^7 \approx 4.1 \times 10^{42}$. Only the first two of these numbers are small enough for a computer to examine every array.

3.3 The smallest case

The double-one set has the three pieces $\boxed{0|0}$, $\boxed{0|1}$, $\boxed{1|1}$, and the 2×3 board has the three tilings shown above. Writing the array 001/011 into them gives



and in each of them the pairs are 00, 01, 11. All three tilings are solutions, so $s(001/011) = 3$ and $M_1 = T_1 = 3$. Some arrays do less well. For 000/111 the first tiling gives the pairs $01, 01, 01$ and is not a solution, while the other two are; so $s(000/111) = 2$.

From $n=2$ on, M_n is smaller than T_n, and the interesting question is how much smaller.

3.4 Symmetries of arrays

The rectangle has four symmetries: the identity, the reflections left–right and top–bottom, and the rotation by 180°. (It is not square, so there is no reflection in a diagonal.) Applying one of them to an array, or renaming the numbers $0, 1, \ldots, n$ by any permutation, gives an array with the same number of solutions. These $4 \cdot (n+1)!$ operations form a group, and two arrays are *essentially the same* if one is obtained from the other by an operation of this group. When I count optimal arrays "up to symmetry", I count classes of essentially the same arrays; for $2 \le n \le 5$ the total number of optimal arrays is also given. Symmetry is used only for grouping arrays: The numbers T_n, $s(A)$, and M_n always count all tilings, without identifying any two of them. The renaming of numbers does not change which blocks are good or bad (Section 4), so when sets of blocks are listed "up to symmetry", only the four symmetries of the rectangle are used.

4 Flips, and good and bad blocks

The idea. Two solutions of the same array usually differ only a little. The smallest possible change is to take two parallel dominoes that form a 2×2 square and turn them by 90°. Whether this turn keeps a solution a solution depends only on the four numbers in the square, and not on the rest of the board. So the 2×2 squares of an array decide almost everything.

Number the rows from 0 at the top and the columns from 0 at the left. A *block* is a 2×2 square of cells. The $(n+1) \times (n+2)$ board has $n(n+1)$ blocks, and we name a block by its top left cell (r, c). A *flip* in a block replaces two horizontal dominoes that fill the block by two vertical ones, or the other way round. A flip changes a tiling into another tiling, and every two tilings of a rectangle are connected by a chain of flips (W. P. Thurston, "Conway's tiling groups," *American Mathematical Monthly* **97** (1990), 757–773).

It is convenient to think of a flip as a link between two tilings: one tiling in which the block is filled by two horizontal dominoes, and the tiling that has two vertical dominoes there instead and agrees with it everywhere else. We say that this flip takes place *in* the block. The *number of flips in a block* is the number of such pairs of tilings, which is the same as the number of tilings in which the block is filled by two horizontal dominoes. Like T_n, it depends only on the board and the block, not on any numbers. A tiling takes part in at most one flip in a given block, because the block is either filled by two parallel dominoes or it is not; so two different flips in the same block never involve the same tiling. (In all the boards below, the four corner blocks have the most flips.)

Suppose a block contains the numbers $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Call the block *good* if $a = d$ or $b = c$, that is, if one of its two diagonals carries two equal numbers; otherwise call it *bad*.

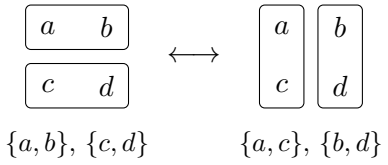


Figure 1: A flip. It keeps the two pieces used if and only if $a = d$ or $b = c$.

Lemma 1. *A flip in a block keeps the multiset of pieces used by a tiling if and only if the block is good.*

Proof. The flip exchanges the pieces $\{a, b\}$ and $\{c, d\}$ for $\{a, c\}$ and $\{b, d\}$ (Figure 1), and leaves all other dominoes alone. If $a = d$, both pairs of pieces are $\{a, b\}$, $\{a, c\}$. If $b = c$, both are $\{a, b\}$, $\{b, d\}$. Conversely, if the two pairs of pieces agree, then $\{a, b\}$ equals $\{a, c\}$, so that $b = c$, or it equals $\{b, d\}$, so that $a = d$. □

Lemma 1 has two consequences, one for good blocks and one for bad blocks, and the whole proof consists of using both of them.

Good blocks. Fix an array A, and let $\mathrm{bad}(A)$ be its set of bad blocks. Join two tilings by an edge when they differ by a flip in a good block of A. The tilings then fall into connected *classes*. By Lemma 1, all tilings of a class use the same multiset of pieces. A tiling is a solution exactly when its multiset of pieces has no repetition, so either every tiling in a class is a solution, or none is.

Corollary. *The solutions of A are a union of whole classes, and the number of solutions is the total size of the classes that consist of solutions.* □

Notice that the classes depend only on the set $\mathrm{bad}(A)$ of bad blocks, and not on the numbers themselves. The numbers only decide which classes are classes of solutions. To have many solutions, an array therefore wants few bad blocks, so that the classes are large; but the numbers must still turn one of the large classes into a class of solutions, and that is not always possible.

Bad blocks. If a block is bad in A, a flip in it changes the multiset of pieces. A solution uses every

piece exactly once. A tiling has exactly as many dominoes as there are pieces, so a tiling with a different multiset of pieces must miss some piece and repeat another, and it is not a solution. Hence:

Lemma 2. *A flip in a bad block never joins two solutions.* □

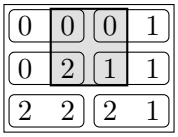
The next section shows these two facts at work on a board small enough to draw completely.

5 The double-two set: the method in one picture

The double-two set has six pieces, 00, 01, 02, 11, 12, 22, and the board is 3×4. It has 11 tilings and 6 blocks. Take the array

$$A_2 = 0001/0211/2221.$$

Its tiling by horizontal dominoes is a solution, because the pairs are 00, 01; 02, 11; 22, 21, all different:



The six blocks of A_2 are

$$(0,0)\colon \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right),\quad (0,1)\colon \left(\begin{smallmatrix}0&0\\2&1\end{smallmatrix}\right),\quad (0,2)\colon \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right),\quad (1,0)\colon \left(\begin{smallmatrix}0&2\\2&2\end{smallmatrix}\right),\quad (1,1)\colon \left(\begin{smallmatrix}2&1\\2&2\end{smallmatrix}\right),\quad (1,2)\colon \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right).$$

Five of them have equal numbers on a diagonal. Only the block $(0,1)$, which contains $\left(\begin{smallmatrix}0&0\\2&1\end{smallmatrix}\right)$, is bad: There $0 \neq 1$ and $0 \neq 2$. It is shaded in the picture above.

Figure 2 shows all 11 tilings of the 3×4 board, filled with the numbers of A_2. Two tilings are joined when they differ by a flip, and the edge is labeled by the block in which the flip takes place. There are 14 flips in all. Exactly one of them takes place in the bad block $(0,1)$: It joins tiling 10 to tiling 11, and it is drawn dashed.

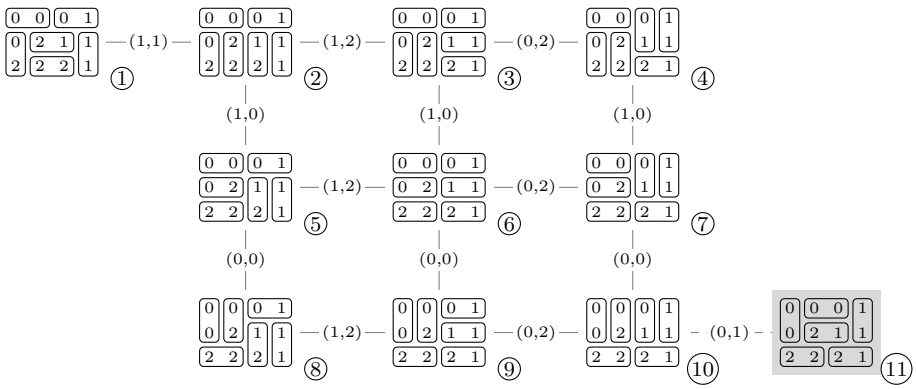


Figure 2: The 11 tilings of the 3×4 board, filled in with the array $A_2 = 0001/0211/2221$, and the 14 flips between them, each labeled by its block; the circled numbers name the tilings. The flip in the bad block $(0,1)$ is dashed. Removing it cuts off tiling 11 (shaded), which uses the piece 00 twice; the other 10 tilings form one class, and they are the solutions of A_2.

Now the Corollary can be read off the picture. If we remove the dashed edge, tiling 11 is left on its own, and the other ten tilings remain connected. So there are two classes, of sizes 10 and 1. The large class contains the horizontal tiling 6, which is a solution, so all ten of its tilings are solutions. Tiling 11 is not: Its two dominoes in the top left corner both carry 0 and 0. Hence A_2 has exactly 10 solutions. (One can also check directly what the dashed flip does: Tiling 10 uses

$\{0,2\}$ and $\{0,1\}$ in the block $(0,1)$, tiling 11 uses $\{0,0\}$ and $\{2,1\}$ there, and since 00 already occurs elsewhere in tiling 11, the flip destroys the solution, as Lemma 2 says it must.)

Why can no array of the double-two set have 11 solutions? There are two cases, and they correspond to the two consequences of Lemma 1.

- *The array has a bad block.* Every block of the 3×4 board carries at least one flip. Indeed, counting the edges with each label in Figure 2, the blocks $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$ have 3, 1, 3, 3, 1, 3 flips. By Lemma 2 the two tilings joined by a flip in the bad block cannot both be solutions, so at most 10 of the 11 tilings are solutions.
- *The array has no bad block.* Then all 11 tilings form a single class, so either all of them are solutions or none is. Here I simply examined all 34,650 arrays with four copies of each number (Section 3). Exactly 234 of them have no bad block, and in each of these 234 some number never occurs in two adjacent cells, so its double cannot be placed and the array has no solution. I know no short proof of this step by hand; it is the one place where a search replaces a deduction, and on the larger boards it becomes the satisfiability test of Section 6.

So $M_2 = 10$. The same examination of all arrays shows that exactly 240 arrays have 10 solutions, 10 of them essentially different. Each has exactly one bad block, always $(0,1)$ or $(1,1)$, one of the two blocks that carry a single flip.

6 The general method

The idea. When an array has many bad blocks, Lemma 2 alone gives a good upper bound. When it has few bad blocks, the classes are few and large, and we can ask, class by class, whether any choice of numbers makes the class a class of solutions. Each of the two arguments alone would fail: The first is too weak when there are few bad blocks, and the second would need far too many tests if it had to be run for every possible set of bad blocks. Together they suffice, because the first argument already disposes of every set of bad blocks except a comparatively short list of small ones, and only the sets on that list need the second argument. For the double-six board, every set of ten or more blocks is disposed of by the first argument, and only 11,325 sets (up to symmetry) remain for the second.

From now on S denotes any set of blocks, and $T = T_n$.

6.1 Many bad blocks: a matching bound

Let H_S be the graph on the T tilings whose edges are the flips in blocks of S.

Lemma 3. *If $S \subseteq \mathrm{bad}(A)$, and H_S has a matching with m edges, then the array A has at most $T - m$ solutions.*

Proof. Each block of S is bad in A, so by Lemma 2 no edge of H_S joins two solutions; in other words, the solutions form an independent set in H_S. An independent set contains at most one end of each edge of the matching, so it misses at least m of the T tilings. □

(The flip graph is bipartite. A flip creates or removes two vertical dominoes in adjacent columns, so it changes by one the number of vertical dominoes that lie in even-numbered columns, and the parity of that number alternates along every path. By Kőnig's theorem, T minus the size of a maximum matching is then exactly the size of a largest independent set of H_S. So Lemma 3 is the strongest bound that can be obtained from the fact that the solutions form an independent set in H_S.)

In the double-two case the bound is at most $T - 1 = 10$ for every nonempty S; it equals 10 for $S = \{(0,1)\}$ and $S = \{(1,1)\}$, and it is 8 for each of the other four single blocks. In general, write

$\nu(H_S)$ for the number of edges in a largest matching of H_S, and with a target M in mind call a set S *dangerous* if $T - \nu(H_S) > M$. If bad(A) is not dangerous, Lemma 3 shows that A has at most M solutions.

If $S' \subseteq S$, then $H_{S'}$ is a subgraph of H_S, so $\nu(H_{S'}) \le \nu(H_S)$; hence a subset of a dangerous set is dangerous. The dangerous sets can therefore be listed level by level: A set of $k+1$ blocks needs to be looked at only if all its subsets of k blocks are dangerous. A symmetry of the rectangle maps blocks to blocks, good blocks to good blocks, and H_S to an isomorphic graph, so it suffices to list the dangerous sets up to symmetry.

6.2 Few bad blocks: which classes can be solutions

Fix a set S, and form the classes of tilings under flips in the blocks *outside* S. Call a collection of these classes *realizable* if there is an array A with bad$(A) \subseteq S$ for which every class in the collection consists of solutions; call it *exactly realizable* if such an array exists with bad$(A) = S$, that is, with $a \neq d$ and $b \neq c$ in every block of S as well. One tiling from each class suffices as a test, because a class consists of solutions as soon as one of its tilings is a solution. So realizability is a finite question about the numbers: Choose a number for each cell so that each block outside S has equal numbers on a diagonal (and, for exact realizability, each block of S has not), and so that the dominoes of each chosen tiling carry different pieces.

Lemma 4. *If* bad$(A) = S$*, the number of solutions of A is at most the largest total size of an exactly realizable collection of classes for S, and hence at most the largest total size of a realizable collection.*

Proof. By the Corollary, the solutions of A are the union of some classes for $S =$ bad(A), and A itself shows that this collection is exactly realizable. □

For a dangerous set S one does not need the exact maximum in Lemma 4, only a proof that it is at most M. Usually it is enough to test the classes one at a time, largest first: If the realizable classes have total size at most M, the set S is settled. When several classes are realizable on their own and their sizes add up to more than M, the following search finds the largest total size of a realizable collection. List the classes that are realizable on their own as $K_1, K_2, \ldots, K_r$, by decreasing size. The search keeps a realizable collection F of classes taken from $K_1, \ldots, K_{i-1}$ and decides, for $i = 1, 2, \ldots$, whether K_i joins F. At each step it explores two branches: first the branch with $F \cup \{K_i\}$, provided this collection passes the realizability test, and then, in every case, the branch with F unchanged. A branch is abandoned when the size of F plus the total size of $K_i, \ldots, K_r$ cannot exceed the best total found so far. Every realizable collection is a set of classes each of which is realizable on its own, and a subset of a realizable collection is realizable; so the search never discards a branch that could lead to a larger realizable collection, and the best total it finds is the true maximum.

Proposition. *If every dangerous set S has been shown to allow at most M solutions in this way, then $M_n \le M$.*

Proof. Let A be any array. If bad(A) is not dangerous, Lemma 3 gives at most M solutions. If bad(A) is dangerous, it is one of the listed sets up to a symmetry, and Lemma 4 (or, for nine sets in Section 10, the refinement described there) gives at most M solutions. □

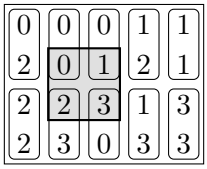
The next three sections apply the Proposition with M equal to the number of solutions of the best array we know. In each case the list of dangerous sets is short enough to print, and the cases $n = 3$ and $n = 4$ can also be checked by an exhaustive search that knows nothing about blocks.

7 The double-three set: 4×5

The board has 95 tilings and 12 blocks. The array

$$A_3 = 00011/20121/22313/23033$$

has 88 solutions. Its tiling by vertical dominoes is one of them (the pairs in the top half are 02, 00, 01, 12, 11, and in the bottom half 22, 23, 03, 13, 33), and it has a single bad block, $(1,1)$, which contains $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right)$:



With only this block bad, the 95 tilings fall into three classes, of sizes 88, 6, and 1, and the class of 88 is the set of solutions.

Now take $M = 88$. The bad block $(1,1)$ carries 6 flips. No two of them involve the same tiling (Section 4), so they form a matching of size 6, and $95 - 6 = 89 > 88$. So the set $\{(1,1)\}$ is dangerous, and so is its mirror image $\{(1,2)\}$. So is the empty set, for which the bound is 95. Every other single block carries at least 11 flips, and for the two blocks $(1,1)$ and $(1,2)$ together the largest matching has 10 edges. So every nonempty set of blocks other than $\{(1,1)\}$ and $\{(1,2)\}$ has a matching with at least 10 edges, its bound is at most 85, and there are no further dangerous sets. Up to symmetry, the two dangerous sets are settled as follows.

- $S = \emptyset$: All 95 tilings form one class, and it is not realizable. No array with all twelve blocks good has a solution.
- $S = \{(1,1)\}$: Of the three classes 88, 6, 1, only the class of 88 is realizable. So an array whose only bad block is $(1,1)$ has at most 88 solutions.

Hence $M_3 = 88$. The argument also shows that an optimal array has exactly one bad block, $(1,1)$ or $(1,2)$: Every other set of bad blocks allows at most 85 solutions, and the empty set allows none.

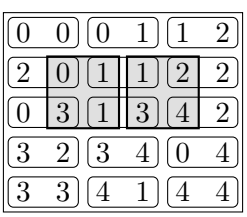
This case is small enough to check in a completely different way. The number of arrays with five copies of each number is $20!/(5!)^4 = 11{,}732{,}745{,}024$, and counting the solutions of all of them confirms that the maximum is 88. It is attained by exactly 1,536 arrays, which are 16 essentially different arrays. The next best count is 83: No array has between 84 and 87 solutions.

8 The double-four set: 5×6

The board has 1,183 tilings and 20 blocks. The array

$$A_4 = 000112/201122/031342/323404/334144$$

has 1,012 solutions. Its tiling by horizontal dominoes is one of them, and its bad blocks are $(1,1)$ and $(1,3)$, which contain $\left(\begin{smallmatrix} 0 & 1 \\ 3 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$:



With these two blocks bad, the tilings fall into seven classes, of sizes 1,012, 60, 60, 21, 14, 14, 2, and the largest class is the set of solutions.

With $M = 1{,}012$ there are 15 dangerous sets, or 6 up to symmetry. Here is the complete list, with the bound of Lemma 3, the sizes of the largest classes, and how each set is settled.

S	$T - \nu(H_S)$	largest classes	settled because
$\emptyset$	1183	1183	no class is realizable
$\{(0,1)\}$	1030	1014, 169	no class is realizable
$\{(1,1)\}$	1103	1086, 81, 16	no class is realizable
$\{(1,1),(1,3)\}$	1051	1012, 60, 60, 21, ...	only the class of 1012 is realizable
$\{(1,1),(2,1)\}$	1053	1021, 65, 65, 16, ...	no class is realizable
$\{(1,1),(2,3)\}$	1035	999, 76, 76, 11, ...	realizable classes total 999

Hence $M_4 = 1{,}012$. This table summarizes the whole proof; besides the realizability tests, it rests on the computed matchings and classes. It is worth noticing how close the method comes to the truth: The empty set has a class of all 1,183 tilings, the set $\{(0,1)\}$ a class of 1,014, and the set $\{(1,1),(2,1)\}$ a class of 1,021, all larger than 1,012, but no choice of numbers makes any of them a class of solutions.

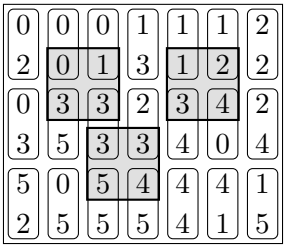
Direct enumeration of all arrays is infeasible here, since there are about 1.4×10^{18} of them. But a branch-and-bound search that fills the array cell by cell, and keeps track of the tilings that are still free of repeated pieces, can find all arrays with more than 900 solutions without using blocks at all. It finds the same maximum 1,012, attained by exactly 120,960 arrays (252 essentially different ones); above 900 only the counts 901, 950, 999, and 1,012 occur.

9 The double-five set: 6×7

The board has 31,529 tilings and 30 blocks. The array

$$A_5 = 0001112/2013122/0332342/3533404/5054441/2555415$$

has 23,929 solutions, among them its tiling by vertical dominoes. It has three bad blocks, $(1,1)$, $(1,4)$, and $(3,2)$:



With these three blocks bad there are 16 classes; the largest has 23,929 tilings and is the set of solutions, and the next ones have 2,776, 1,524, and 1,244.

With $M = 23{,}929$ there are 416 dangerous sets, or 115 up to symmetry.

size of S	0	1	2	3	4	5	total
dangerous sets (up to symmetry)	1	7	35	48	21	3	115
no class realizable	1	7	35	40	7	0	90
realizable classes total $\leq M$	0	0	0	8	7	0	15
branch and bound needed	0	0	0	0	7	3	10

In the ten sets that need the branch-and-bound search, the largest total sizes of realizable collections (in the sense $\text{bad}(A) \subseteq S$) are 22,842, 23,189 (twice), 23,650 (twice), and 23,929 (five

times). The five sets that reach 23,929 all contain an image of the three bad blocks of A_5, and the arrays that realize the value have exactly those three bad blocks; the extra blocks of S are good in them. Hence $M_5 = 23{,}929$. All 101,341,440 optimal arrays (35,188 essentially different ones) have the same three bad blocks, up to symmetry.

The branch-and-bound search described in Section 8, which does not use blocks at all, can still be run to the end on this board. It finds no array with more than 23,929 solutions, and it finds the same 101,341,440 optimal arrays. So for $n = 5$, as for $n \le 4$, the maximum is obtained by two independent methods.

10 The double-six set: Dominosa itself

Now we return to the real game. The board has 1,292,697 tilings and 42 blocks, and the record array A^* has only four bad blocks, namely $(1,3)$, $(2,1)$, $(2,5)$, and $(4,3)$, arranged as a kite that is symmetric under left–right reflection (Figure 3). With these four blocks bad, the tilings fall into 34 classes. The largest class has 853,985 tilings, the next ones have 95,456, 95,456, 60,344, and 39,800. The solutions of A^* are exactly the largest class. For example, the tiling by horizontal dominoes in Figure 3 is a solution, and it can be flipped in the good block $(0,0)$, which contains $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$; the pieces 00 and 02 become 02 and 00.

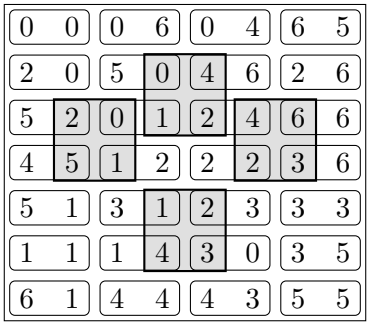


Figure 3: The array A^* with 853,985 solutions. Its four bad blocks are shaded. The 28 horizontal dominoes drawn here form one solution: The pairs 00, 06, 04, 56; 02, 05, 46, 26; ... are all different.

The earlier records. The bad blocks of González-Morris's array are the same four blocks and one more, the block $(4,5)$. Keller's array of answer 73 has seven bad blocks: those four and $(1,1)$, $(4,1)$, $(4,5)$ (Figure 4). The extra bad block $(4,5)$ cuts the class of size 853,985 into pieces of sizes 807,752, 28,239, and 17,994, and González-Morris's solutions are the first piece. So the step from 807,752 to 853,985 is the step from five bad blocks to four.

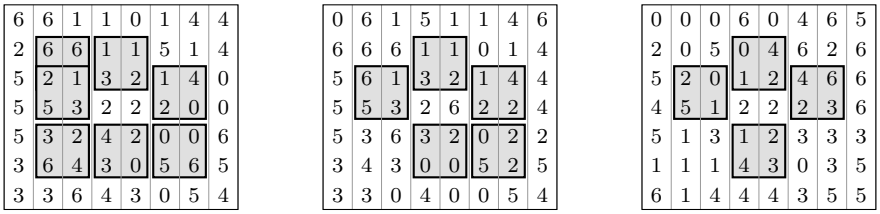


Figure 4: The bad blocks of Keller's array, of González-Morris's array, and of A^*, with their numbers of solutions.

The upper bound. With $M = 853{,}985$ there are 11,325 dangerous sets up to symmetry, including the empty set. No dangerous set has ten or more blocks:

size of S	0	1	2	3	4	5	6	7	8	9	total
dangerous sets	1	12	197	1,290	3,319	3,774	2,109	534	88	1	11,325

They are settled as in the smaller cases.

- For 10,823 of them, the empty set among them, the classes that are realizable on their own have total size at most 853,985. (None of the 11,325 dangerous sets has a realizable class of more than 853,985 tilings.)
- For 493 sets several classes are realizable on their own, and their sizes add up to more than 853,985. The branch-and-bound search over collections finds the exact maximum of Lemma 4 for each of them, and it is at most 853,985 in every case.
- The last 9 sets, of eight or nine blocks, are settled by a refinement of Lemma 3. Let A be an array with $\mathrm{bad}(A) = S$, and let R be the union of the classes that are realizable on their own. Every solution of A lies in R, and since every block of S is bad in A, no flip in a block of S joins two solutions. So A has at most $|R| - m$ solutions, where m is the size of a matching of H_S whose edges join tilings in R. These bounds lie between 718,025 and 780,001.

By the Proposition, no array has more than 853,985 solutions, and A^* attains this number. This proves the Theorem. □

Section 11 describes how these computations were done, and gives intermediate numbers against which another implementation can be compared.

11 How the computations were done

This section describes the computations behind Sections 5–10 in enough detail that they can be repeated independently; the programs, with instructions for building and running them and the expected outputs, are at

`https://github.com/daizisheng/dominosa-max`.

The upper bound for $n = 6$ was computed twice, by two separately written sets of programs that share no code, and the step that finds the classes was checked a third time by a different method (Section 11.2).

11.1 Tilings, flips, and matchings

The tilings of the $(n+1) \times (n+2)$ board are listed by filling the cells in row-major order, each still empty cell being covered either by a horizontal domino or by a vertical one; for $n = 6$ this gives the 1,292,697 tilings of answer 72. For each tiling and each block filled by two horizontal dominoes, the flipped tiling is found by table lookup. For $n = 6$ there are 8,828,108 flips in all. The number of flips in a single block ranges from 85,612 for the block $(1, 1)$ to 388,705 for the corner block $(0, 0)$; the twelve values for the blocks (r, c) with $r \le 2$ and $c \le 3$, which represent all 42 blocks up to symmetry, are

	$c = 0$	$c = 1$	$c = 2$	$c = 3$
$r = 0$	388,705	183,779	306,167	203,435
$r = 1$	228,897	85,612	175,534	92,686
$r = 2$	287,253	115,368	224,589	126,125

For a single block b the flips in b form a matching, so $\nu(H_{\{b\}})$ is the number in this table. For larger sets S the maximum matching $\nu(H_S)$ was computed by the augmenting-path method of Hopcroft and Karp in the first computation, and by a simpler augmenting-path method, component by component of H_S, in the second. In the second computation every matching that was used to show that a set is *not* dangerous (9,530 sets up to symmetry) was checked separately: Each edge was verified to be a flip in a block of S, and no tiling was allowed to appear twice.

The dangerous sets were listed level by level as in Section 6, each set being represented by the smallest of its four images under the symmetries of the rectangle. Both computations produced the same 11,325 sets, with the level counts shown in Section 10.

11.2 Classes

For each dangerous set S the classes, that is, the connected components of the graph of flips in blocks outside S, were found by union–find in the first computation and by breadth-first search in the second.

A third way to find the classes avoids listing the tilings altogether. It rests on Thurston's height function, which assigns an integer to every grid point of the board, determined by the tiling, such that a flip in a block changes the height only at the center of that block, by ± 4, and the heights at the boundary are the same for all tilings. Along a vertical grid line, the height at each point depends only on which rows have a horizontal domino crossing that line.

Lemma 5. *For any set S of blocks, the classes under flips in blocks outside S are exactly the nonempty sets of tilings on which the heights at the centers of the blocks of S take prescribed values.*

Proof sketch. Flips outside S do not change the heights at the centers of the blocks of S. Conversely, the tilings with prescribed heights at these centers form a set of height functions closed under pointwise minimum, so it has a least member $h_{\min}$. If h is any other member, a grid point v where $h(v) > h_{\min}(v)$ and $h(v)$ is as large as possible is not the center of a block of S, and it is a local maximum of h: Along an edge vw both h and $h_{\min}$ change by $+1$ or -3 (or by -1 or $+3$, depending on the edge), and heights at a grid point agree modulo 4; so a neighbor w with $h(w) > h(v)$ would satisfy $h(w) > h_{\min}(w)$, contradicting the choice of v. An interior local maximum is the center of a block filled by two parallel dominoes, so that block can be flipped, lowering $h(v)$ by 4 without going below $h_{\min}$. (These facts about height functions are standard; see W. P. Thurston, loc. cit., and E. Rémila, "The lattice structure of the set of domino tilings of a polygon," *Theoretical Computer Science* **322** (2004), 409–422.) Repeating this connects h to $h_{\min}$ by flips outside S. □

By Lemma 5, the class sizes for S can be computed by a transfer matrix that runs over the columns of the board and keeps, besides the set of rows crossed by horizontal dominoes, the heights reached so far at the centers of the blocks of S. This takes a few milliseconds per set. Computed in this way, the class sizes of all 11,325 dangerous sets, 572,904 classes in all, agree with those of the second computation, and the twelve numbers in the table above are recovered as well.

11.3 Realizability tests

A realizability test asks for numbers in the cells such that every block outside S is good (and, in the exact version, every block of S is bad), and such that each of the given tilings is a solution. Both computations encode this as a satisfiability problem with these variables and conditions:

- one variable for each cell and each number, with exactly one number per cell;
- for each block outside S, auxiliary variables expressing "$a = d = k$" and "$b = c = k$" for each number k, and a clause requiring one of them;

- for each domino of each given tiling and each piece, a variable saying that the domino carries that piece, with each domino carrying one piece and each piece carried by at most one domino of the tiling;
- a condition that removes the freedom of renaming the numbers: In the first computation the number in the top left cell is fixed to be 0; in the second, the seven doubles are required to occur in a fixed order along the first given tiling.

The first computation tested realizability and the second exact realizability (Lemma 4 allows either). The two computations used different encodings of the last two conditions and different solvers. In the first computation, the tests showing that no single class of more than 853,985 tilings is realizable were repeated with a second encoding and a second solver, and so were a random sample of the other tests. In the second computation each unsatisfiable answer was accompanied by a proof in the DRAT format, which was checked by an independent proof checker; each satisfiable answer was turned into an array whose solutions were then counted directly. The encoding only forces the requested classes to consist of solutions, so the count must be at least their total size; in every case it was exactly that total. The second computation made 15,296 tests, of which 13,786 were unsatisfiable and 1,510 satisfiable; 10,111 of the dangerous sets needed only one test.

As a check that the encodings do not forbid too much, realizations of 220 known solvable instances were requested: arrays obtained by placing the pieces on a random tiling, with that tiling as the known solution, and the record arrays under random symmetries and renamings, and all of them were found.

11.4 Numbers for comparison

Besides the counts of dangerous sets by level, the following outcomes of the second computation can be compared with any other implementation. The largest total size of an exactly realizable collection is 853,985, and it occurs only for the bad blocks of A^* (up to symmetry). The next largest totals that were found are 846,461, 839,062, 834,082, 828,810, and 826,452; an array with 846,461 solutions is 00211105/02221510/25233103/42335360/14343626/44456166/04555646. As a control, the whole computation was repeated with the target 853,984 in place of 853,985. Then exactly one more set becomes dangerous, and exactly one set is reported as exceeding the target, namely the set of bad blocks of A^*.

11.5 The smaller boards

For $n = 2$ and $n = 3$ all arrays with $n+2$ copies of each number were examined by two separately written programs, which agree on the number of arrays with each number of solutions. For $n = 4$ and $n = 5$ the method of Section 6 was used, with every unsatisfiable test confirmed by a second encoding and solver. Independently, a branch-and-bound search over arrays was run to completion: It fills the cells one at a time, keeps the set of tilings whose completed dominoes carry no repeated piece, and abandons a partial array as soon as fewer tilings than the target remain. It uses no blocks, flips, classes, or satisfiability tests, and it finds the same maxima and the same optimal arrays.

11.6 Running time

All computations were run on a machine with a 32-thread Intel Xeon Platinum 8375C processor at 2.9 GHz and 123 GB of memory. The first computation of the upper bound for $n = 6$ took about 37 core-hours, including the search for the record; the second took about 20 core-hours, mostly in the satisfiability tests and the checking of their proofs. The transfer-matrix computation of all

classes takes a few minutes. Counting the solutions of A^* takes a fraction of a second. For the smaller boards, the exhaustive search for $n = 3$ took about half a core-hour, and the branch-and-bound search over arrays for $n = 5$ about 16 core-hours.

12 The numbers

The results for all six sets are collected below. For $n \leq 5$ the column "bad blocks" gives the number of bad blocks, which is the same for every optimal array, and the last column gives the number of optimal arrays up to the group of $4 \cdot (n+1)!$ symmetries and renamings of Section 3; these classifications come from the computations of Sections 5–9. For $n = 6$ the bad blocks are those of A^*, and the last entry is only the number of essentially different optimal arrays that I found.

n	board	T_n	M_n	M_n/T_n	bad blocks	optimal arrays up to symmetry
1	2×3	3	3	1.000	0	1
2	3×4	11	10	0.909	1	10
3	4×5	95	88	0.926	1	16
4	5×6	1,183	1,012	0.855	2	252
5	6×7	31,529	23,929	0.759	3	35,188
6	7×8	1,292,697	853,985	0.661	4	$\geq$ 19,601

A few features of the table stand out.

- In every case from $n = 1$ to $n = 5$, the solutions of an optimal array form a single class, and the tiling by horizontal dominoes (for even n) or by vertical dominoes (for odd n) is one of them. The record array A^* for $n = 6$ behaves in the same way.
- The number of bad blocks of an optimal array grows slowly, 0, 1, 1, 2, 3, 4, while the number of blocks grows like n^2. An optimal array is thus very close to an array with no bad block at all; but an array with no bad block has no solution, for every n from 2 to 6, because its single class of all tilings is never realizable.
- For n from 2 to 5, no optimal array is fixed by a nontrivial symmetry combined with a renaming, so the total number of optimal arrays is $4 \cdot (n+1)!$ times the number in the last column: 240, 1,536, 120,960, and 101,341,440.

The sequence $3, 10, 88, 1{,}012, 23{,}929, 853{,}985$ is not in the OEIS (searched on 18 November 2026), and I did not find it elsewhere in the literature, nor any upper bound for the double-six case. The question of the largest number of solutions is also posed on Michael Keller's page "Puzzle Laboratory Guide to Domino Logic Puzzles" (last edited 10 June 2024, consulted on 18 November 2026),

`https://solitairelaboratory.com/puzzlelaboratory/DominoGG.html`,

which lists the records 52,224 (Harry Nelson and John Fletcher), 730,924, and 807,752, without an upper bound.